How Often Is a Random Integer Matrix Singular?

Exact counts for 2×2, a proved asymptotic with explicit constant for 3×3, a numerically identified 4×4 constant, and a conjectured second-order term

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Abstract

Fill an $n \times n$ matrix with independent entries drawn uniformly from $\{1, \dots, N\}$. How likely is it to be singular? For continuous entries the answer is zero, and for fixed N and growing n it is governed by deep results of Kahn–Komlós–Szemerédi, Tao–Vu and Tikhomirov. Here we study the opposite regime, fixed n and growing N. For $n = 2$ we recall the exact formula $S_2(N) = 2\sum_{m \le N} \varphi(m) \lfloor N/m \rfloor^2 - N^2$ for the number of singular matrices, which is equivalent to a gcd-sum formula already recorded in the OEIS. For $n = 3$ we prove an exact decomposition of the count $S_3(N)$ into a parallel-pair term and an *axis-plane* term, both in closed arithmetic form and asymptotic to κN^6 and $3\kappa N^6$ with $\kappa = 2\zeta(2)/\zeta(3) - 1$, plus a *generic-plane* term that carries the difficulty. Following Katznelson's plane-by-plane strategy, we then prove by elementary means that

$$S_3(N) = C_3\, N^6 \log N + O(N^6), \qquad C_3 = \frac{15\,(12 - \pi^2)}{4\,\zeta(3)} \approx 6.6461,$$

so that $\Pr[\det M = 0] = C_3 \log N / N^3 + O(N^{-3})$. The constant is an integral over plane sections of the unit cube, which we compute in closed form. We prove that this integral equals twice the density at 0 of the determinant of a random real matrix with independent uniform $[0, 1]$ entries. In every dimension, the same method predicts the leading constant $C_n = (n-1)\, p_n(0)/\zeta(n)$. For $n = 4$ we reduce the constant to one-dimensional integrals, evaluate them to more than 100 digits, and identify, numerically and without proof, the closed form

$$J_4 = \tfrac{1387213}{1458} - \tfrac{736}{243}\pi^2 - \tfrac{101434}{27}\zeta(3) + \tfrac{421696}{81}\log 2,$$

giving the predicted 4×4 constant $C_4 = 135 J_4/\pi^4 = 19.7640\ldots$ A new $O(N^6)$ algorithm gives the exact counts $S_3(N)$ for all $N \le 100$, which appear to be new. Finally, we study the second-order term. For $n = 2$ it is explicit, $c_2 = \frac{12}{\pi^2}(2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}) - 2$. For $n = 3$ we conjecture a formula for $c_3 = \lim_N (S_3(N)/N^6 - C_3 \log N)$ in terms of a lattice-sum constant and an average over random lattices. The same formula reproduces c_2 exactly in dimension two, and it gives $c_3 = -2.7381$, consistent with direct computations up to $N = 10^6$. In short, the results for $n = 2$ and $n = 3$ are theorems, except the value of c_3, which is a conjecture. The closed form of J_4 is a numerical identification, and the 4×4 asymptotic is a prediction.

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1 Introduction

A matrix is invertible exactly when its rows are linearly independent, i.e. when $\det M \neq 0$. A natural first question about "random matrices" is therefore: *if the entries are chosen at random, what is the probability that* $\det M = 0$*?*

The answer depends entirely on where the entries come from.

- **Continuous entries.** If the entries are independent with any distribution that has a density (uniform on $[0, 1]$, Gaussian, ...), then M is invertible with probability 1. The singular matrices form the zero set of the polynomial det, a hypersurface of Lebesgue measure zero in $\mathbb{R}^{n^2}$.

- **Finite fields.** Over $\mathbb{F}_q$ the classical count gives $\Pr[\text{invertible}] = \prod_{k=1}^{n}(1-q^{-k})$, which tends to a positive constant (≈ 0.289 for $q=2$) rather than to 1.
- **Small integer alphabets.** For $\{0,1\}$ or $\{\pm 1\}$ entries, the singularity probability is positive and decays exponentially in n. This is a celebrated line of work [16, 11, 22, 23, 10].
- **Integer entries from a growing range.** If the entries are uniform on $\{1,\dots,N\}$ with n fixed, the singularity probability tends to 0 as $N \to \infty$, but never vanishes. This is the regime we study.

Table 1 illustrates how strongly the answer depends on the alphabet, even at $n = 10$.

Entries of a 10×10 matrix	Pr[invertible]	Method
any continuous distribution	1	exact
$\{0,1\}$ over $\mathbb{F}_2$	$\prod_{k=1}^{10}(1-2^{-k}) \approx 0.289$	exact
$\{\pm 1\}$	≈ 0.627	Monte Carlo, 2×10^5 samples
$\{0,1\}$	≈ 0.704	Monte Carlo, 2×10^5 samples
$\{1,2\}$	≈ 0.788	Monte Carlo, 2×10^5 samples
$\{1,2,3\}$	≈ 0.9940	Monte Carlo, 2×10^5 samples
$\{1,\dots,5\}$	≈ 0.99998	Monte Carlo, 2×10^5 samples

Table 1: Invertibility of a random 10×10 matrix for several entry distributions.

Setup. Throughout, $N \geq 1$ is an integer, and we write

$$S_n(N) \;=\; \#\{M \in \{1,\dots,N\}^{n\times n} :\; \det M = 0\}, \qquad \Pr_N[\det M = 0] = \frac{S_n(N)}{N^{n^2}}.$$

There is no uniform distribution on all of $\mathbb{N}$, so "uniform on $\{1,\dots,N\}$, then let $N \to \infty$" is the natural way to make sense of "random natural-number entries".

What is known. Katznelson [13, Thm. 1] proved that for every bounded convex open set $\mathcal{K} \subset \mathbb{R}^{n\times n}$ containing the origin (we follow the formulation in [2]), the number of singular integer matrices in the dilate $T\mathcal{K}$ is $c_{\mathcal{K}}\, T^{n^2-n}\log T + O(T^{n^2-n})$. Wigman [25] made the constant explicit when the rows are bounded in Euclidean length and are required to be primitive. For $n = 2$ and entries in $\{1,\dots,N\}$, the sequence $S_2(N)$ is OEIS A134506 [17], and Shi [20] proved $S_2(N) = \frac{12}{\pi^2}N^2 \log N + cN^2 + O(N^e)$ for every $e > 547/416$. The constant c is in fact explicit: it follows from Bordellès' asymptotic for the gcd-sum function [5], as we explain in Section 2. The companion entry A059306 (entries in $\{0,\dots,N\}$) records an exact gcd-sum formula due to D. Radcliffe (2025), which is equivalent to our Proposition 2.1. Katznelson's method has since been extended to primitive rows [25], to rectangular matrices of fixed rank [14, 9], and to general lattices in $M_n(\mathbb{R})$ [15]. The regime where n and the entry range grow together is treated in [12].

Literature check. Before writing this note we searched the OEIS, arXiv and the wider literature for each result claimed below. The 2×2 formula is known in the equivalent form just mentioned. We found no explicit value of Katznelson's constant for the cube $\{1,\dots,N\}^{n\times n}$, and no OEIS entry or other source for the counts $S_3(N)$, nor for their analogue with entries in

$\{0, \dots, N\}$, which begins $338, 6891, 49246, 228737$. We could not query the OEIS's own search engine directly or consult the full text of [13], so these absences should be confirmed, for instance by submitting the sequence to the OEIS. As an independent check of our counting code, it reproduces the known count of 7875 singular 3×3 matrices with entries in $\{-1, 0, 1\}$ (OEIS A057981; see also [24]). We also found no published value for the density of the determinant of a random 3×3 or 4×4 matrix with uniform entries, which we obtain in Sections 4.5 and 4.6.

What this note adds.

(i) A short geometric proof of an exact formula for $S_2(N)$ in terms of Euler's totient (Proposition 2.1). The formula itself is known in an equivalent gcd-sum form. What we add is the line-by-line picture, which generalises to higher dimensions, and the explicit second-order constant $c_2 = \frac{12}{\pi^2}\big(2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\big) - 2 = -0.51131744\dots$ (Corollary 2.4).

(ii) An exact decomposition of $S_3(N)$ (Theorem 3.6) into *parallel-pair* and *axis-plane* terms, which have closed arithmetic forms, and a *generic-plane* term, which carries the difficulty. The explicit terms are asymptotic to κN^6 and $3\kappa N^6$ with $\kappa = 2\zeta(2)/\zeta(3) - 1$ (Proposition 3.7).

(iii) A proof of the asymptotic, with explicit leading constant, for entries in $\{1, \dots, N\}$ (Theorem 4.1):

$$\boxed{S_3(N) \sim C_3\, N^6 \log N, \qquad C_3 = \frac{15(12 - \pi^2)}{4\,\zeta(3)} = 6.64609426967\dots} \tag{1}$$

The constant is the value of a geometric integral over plane sections of the unit cube, which we evaluate in closed form (Theorem 4.7).

(iv) A probabilistic interpretation (Proposition 4.9): this integral is twice the density at 0 of the determinant of a random real matrix with independent uniform $[0, 1]$ entries. Hence $C_n = (n-1)\,p_n(0)/\zeta(n)$, and the 3×3 determinant density at 0 is $p_3(0) = \frac{45}{2} - \frac{15\pi^2}{8}$.

(v) A closed form for the predicted 4×4 constant (Numerical identification 4.12 and Prediction 4.13), identified numerically from a computation to more than 100 digits, but not proved:

$$C_4 = \frac{135}{\pi^4}\Big(\frac{1387213}{1458} - \frac{736}{243}\pi^2 - \frac{101434}{27}\zeta(3) + \frac{421696}{81}\log 2\Big) = 19.7640630589\dots$$

(vi) An $O(N^6)$ exact algorithm that computes $S_3(N)$ for all $N \le N_{\max}$ at once, and the values for $N \le 100$ (Appendix A), which we did not find in the OEIS or elsewhere.

(vii) A conjectured formula for the second-order constant $c_3 = \lim_N (S_3(N)/N^6 - C_3 \log N)$ (Conjecture 6.1). It is supported by an exact test in dimension two, by a direct check of the equidistribution it relies on, and by computations of $S_3(N)/N^6 - C_3 \log N$ up to $N = 10^6$. Its value is

$$c_3 = -2.7381 \pm 0.0006.$$

Small-N data alone are misleading here, because the convergence is slow: $c_3(N)$ is still -3.28 at $N = 100$ and -2.93 at $N = 1000$.

Status of the results. Table 2 separates what is proved from what is not. Everything stated for 2×2 and 3×3 matrices is proved, except the value of c_3. The proof of the main theorem (Section 5) follows Katznelson's plane-by-plane strategy. It also shows that the position of the origin at a corner of the cube, which puts our region outside the hypotheses of Katznelson's theorem, causes no difficulty.

Result	Status	Where
Exact formula for $S_2(N)$, explicit c_2	theorem	Proposition 2.1 and Corollary 2.4
Exact decomposition of $S_3(N)$	theorem	Theorem 3.6
$S_3(N) = C_3 N^6 \log N + O(N^6)$, with C_3 in closed form	theorem	Theorems 4.1 and 4.7
$J_n = 2p_n(0)$ for every n	theorem	Proposition 4.9
Values of $S_3(N)$ for $N \le 100$	exact computation	Appendix A
$S_n(N) \sim C_n N^{n^2-n} \log N$ for $n \ge 4$	heuristic prediction	(2)
Closed form of J_4, equivalently of $p_4(0)$	numerical identification	Numerical identification 4.12
$C_4 = 19.7640\ldots$	prediction	Prediction 4.13
$c_3 = -2.7381 \pm 0.0006$	conjecture	Conjecture 6.1

Table 2: Status of the main results. A *numerical identification* is an exact formula found by an integer-relation search and confirmed to high precision, but not proved.

Figure 1 shows the exact singularity probabilities for $n = 2$ and $n = 3$ against the leading-order predictions.

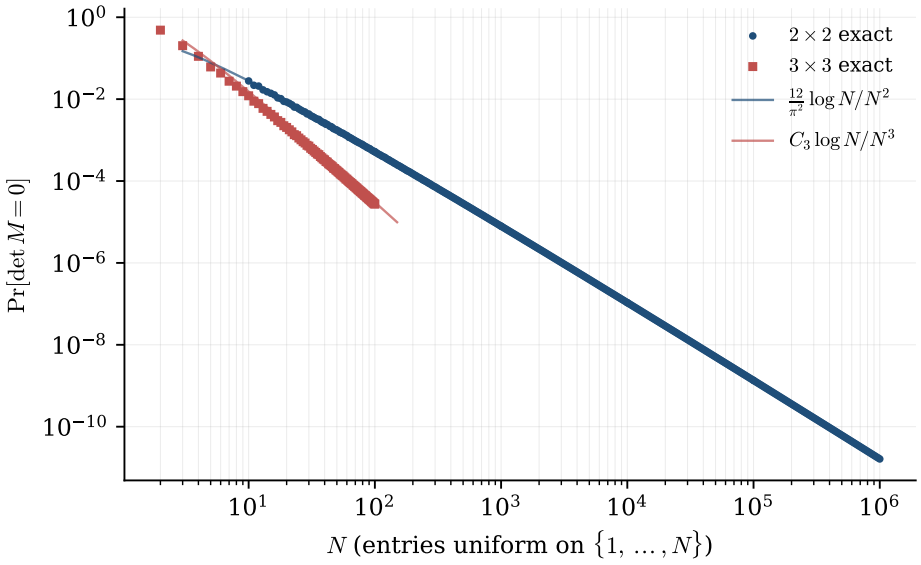


Figure 1: Exact probability that a random matrix with entries uniform on $\{1, \ldots, N\}$ is singular, for 2×2 ($N \le 10^6$) and 3×3 ($N \le 100$) matrices, together with the leading-order asymptotics $\frac{12}{\pi^2} \log N/N^2$ and $C_3 \log N/N^3$. The slopes on the log–log scale reflect the decay rates N^{-2} and N^{-3}. The data sit slightly below the leading terms because the second-order constants c_2 and c_3 are negative (Corollary 2.4 and Conjecture 6.1).

2 Warm-up: the 2×2 case

The 2×2 case already contains the main idea: *group singular matrices by the line (or plane) that contains their rows.*

Proposition 2.1. *For every $N \geq 1$,*

$$S_2(N) \;=\; 2 \sum_{m=1}^{N} \varphi(m) \left\lfloor \frac{N}{m} \right\rfloor^2 \;-\; N^2,$$

where φ is Euler's totient function.

Proof. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with positive entries is singular if and only if its two columns (a, c) and (b, d) are parallel, i.e. both are positive integer multiples of the same primitive vector (p, q), where $\gcd(p, q) = 1$. The multiples of (p, q) lying in $\{1, \dots, N\}^2$ are $k(p, q)$ for $1 \leq k \leq \lfloor N / \max(p, q) \rfloor$. Each column can be chosen independently, so the direction (p, q) accounts for $\lfloor N / \max(p, q) \rfloor^2$ singular matrices, and distinct directions give disjoint sets.

It remains to count primitive $(p, q) \in \mathbb{Z}^2_{>0}$ with $\max(p, q) = m$. For $m = 1$ there is exactly one, namely $(1, 1)$. For $m \geq 2$ there are $2\varphi(m)$: either $p = m$ and $q < m$ is coprime to m, or the reverse. Hence $S_2(N) = \lfloor N \rfloor^2 + \sum_{m \geq 2} 2\varphi(m) \lfloor N/m \rfloor^2$, which is the stated formula. □

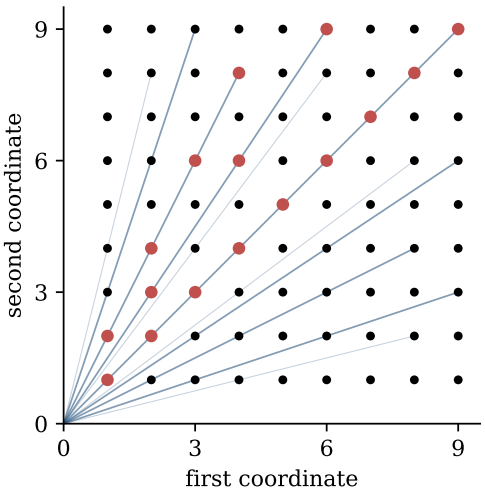


Figure 2: The proof of Proposition 2.1 for $N = 9$. Every point of the grid $\{1, \dots, 9\}^2$ lies on exactly one ray through the origin with a primitive direction (p, q). A singular 2×2 matrix is a choice of two (possibly equal) grid points on a common ray. The points on the rays through $(1, 1)$, $(1, 2)$ and $(2, 3)$ are highlighted.

The formula agrees with OEIS A134506 $(1, 6, 15, 32, 49, 86, 111, \dots)$. We checked it by brute force for $N \leq 12$ and against all 47 terms displayed on the OEIS entry.

Remark 2.2 (Relation to the OEIS gcd-sum formula)**.** Since $\gcd(i, j) = \sum_{d \mid \gcd(i,j)} \varphi(d)$, we have $\sum_{i,j \leq N} \gcd(i, j) = \sum_{d \leq N} \varphi(d) \lfloor N/d \rfloor^2$, so Proposition 2.1 can be rewritten as

$$S_2(N) = 2 \sum_{i,j=1}^{N} \gcd(i, j) - N^2.$$

This is equivalent to the formula $a(n) = 4\sum_{1\le i\le j\le n}\gcd(i,j)+2n^2+3n+1$ recorded by D. Radcliffe (2025) for OEIS A059306, via the identity $\mathrm{A059306}(n) = \mathrm{A134506}(n) + (2n+1)^2$ stated there. We checked numerically that the two agree. Our contribution in this section is therefore only the geometric proof, which is the template for Section 3.

Corollary 2.3. $S_2(N) = \frac{12}{\pi^2}N^2\log N + O(N^2)$. *Equivalently,* $\Pr[\det M = 0] \sim \frac{12}{\pi^2}\,\frac{\log N}{N^2}$.

Proof. Write $\lfloor N/m\rfloor^2 = N^2/m^2 + O(N/m)$ and use $\sum_{m\le N}\varphi(m)/m^2 = \frac{6}{\pi^2}\log N + O(1)$ and $\sum_{m\le N}\varphi(m)/m = O(N)$. □

Shi's theorem [20] sharpens the error term to $cN^2 + O(N^{1.315})$. The constant c is explicit, because Remark 2.2 turns S_2 into a gcd-sum.

Corollary 2.4 (Second-order term for 2×2)**.** *For every $\varepsilon > 0$,*

$$S_2(N) = \frac{12}{\pi^2}N^2\log N + c_2N^2 + O_\varepsilon(N^{1+\theta+\varepsilon}), \qquad \theta \le \tfrac{131}{416},$$

where

$$c_2 = \frac{12}{\pi^2}\Big(2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\Big) - 2 = \frac{12}{\pi^2}\Big(\gamma - \frac{1}{2} + \log\frac{A^{12}}{2\pi}\Big) - 2 = -0.51131744722\ldots,$$

γ is Euler's constant and A is the Glaisher–Kinkelin constant.

Proof. Let $g(n) = \sum_{k\le n}\gcd(k,n)$ be Pillai's gcd-sum function [7]. Splitting $\sum_{i,j\le N}\gcd(i,j)$ into the diagonal and the two triangles gives $\sum_{i,j\le N}\gcd(i,j) = 2\sum_{n\le N}g(n) - \frac{N(N+1)}{2}$. So, by Remark 2.2,

$$S_2(N) = 4\sum_{n\le N}g(n) - 2N^2 - N.$$

Bordellès [5] proved that $\sum_{n\le x}g(n) = \frac{x^2}{2\zeta(2)}\big(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\big) + O_\varepsilon(x^{1+\theta+\varepsilon})$, where θ is the exponent in the Dirichlet divisor problem; Huxley's bound gives $\theta \le 131/416$. Substituting gives the claim, and the second form of c_2 uses $\zeta'(2)/\zeta(2) = \gamma + \log(2\pi) - 12\log A$. □

The exponent $1+\theta = 547/416$ is the one in Shi's theorem. Numerically, $S_2(N)/N^2 - \frac{12}{\pi^2}\log N$ equals -0.49906, -0.50825, -0.51117 and -0.51113 at $N = 10^3, 10^4, 10^5, 10^6$ (Fig. 6(c)).

Remark 2.5 (A correction to the OEIS)**.** Since $\mathrm{A059306}(n) = S_2(n) + (2n+1)^2$, the second-order constant of A059306 is $c_2 + 4 = 3.48868\ldots$. The value $4.5113\ldots$ stated in that entry equals $4 - c_2$, which suggests a sign slip.

The point of Corollary 2.3 for us is the mechanism. A ray with primitive direction of size m contains about N/m grid points and contributes about $(N/m)^2$ matrices. There are about m such directions at scale m, so each dyadic range of m contributes about N^2, and summing over the $\log N$ ranges gives $N^2\log N$. The same mechanism drives the 3×3 count, with lines replaced by planes.

3 An exact decomposition for 3×3

3.1 Notation

Let $\mathcal{B}_N = \{1, \dots, N\}^3$ be the set of possible rows. For a primitive vector $v \in \mathbb{Z}^3_{>0}$, the multiples of v in $\mathcal{B}_N$ are kv with $1 \le k \le k_N(v) := \lfloor N / \|v\|_\infty \rfloor$. For a plane P through the origin, set

$$L_P = \#(P \cap \mathcal{B}_N), \qquad I_P = \#\{(r_1, r_2) \in (P \cap \mathcal{B}_N)^2 : \ r_1 \nparallel r_2\} = L_P^2 - \sum_{\ell \subset P} k_N(\ell)^2,$$

where the sum runs over the lines $\ell \subset P$ through the origin that meet $\mathcal{B}_N$, and $k_N(\ell) = k_N(v)$ for the primitive direction v of ℓ.

3.2 Splitting by the first two rows

Lemma 3.1.

$$S_3(N) = N^3 A(N) + \sum_P L_P \, I_P, \qquad A(N) = \#\{(r_1, r_2) \in \mathcal{B}_N^2 : r_1 \parallel r_2\},$$

where P ranges over planes through the origin. Only finitely many terms are nonzero.

Proof. Fix the first two rows r_1, r_2. If they are parallel, every third row r_3 gives a singular matrix: N^3 choices. Otherwise r_1, r_2 span a unique plane P, and the matrix is singular exactly when $r_3 \in P \cap \mathcal{B}_N$: L_P choices. Grouping the non-parallel pairs by the plane they span gives $\sum_P L_P I_P$. □

The parallel-pair term is completely explicit.

Lemma 3.2. $A(N) = \sum_{m=1}^{N} g(m) \lfloor N/m \rfloor^2$, *where*

$$g(m) = \#\{v \in \mathbb{Z}^3_{>0} \text{ primitive} : \ \|v\|_\infty = m\} = \sum_{d \mid m} \mu(d) \Big[\big(\tfrac{m}{d}\big)^3 - \big(\tfrac{m}{d} - 1\big)^3 \Big].$$

Proof. The first identity is the 3-dimensional version of the proof of Proposition 2.1. For the second, every $x \in \{1, \dots, m\}^3$ with $\|x\|_\infty = m$ is uniquely $d \cdot v$ with v primitive and $\|v\|_\infty = m/d$, so $m^3 - (m-1)^3 = \sum_{d \mid m} g(m/d)$. Möbius inversion finishes the proof. □

Remark 3.3. The same inversion in dimension 2 gives $\sum_{d \mid m} \mu(d) [(m/d)^2 - (m/d - 1)^2] = 2\varphi(m)$ for $m \ge 2$, recovering the totients of Proposition 2.1.

3.3 Which planes occur

Lemma 3.4. *Let P be a plane through 0 containing two non-parallel points of $\mathcal{B}_N$, and let w be a primitive normal vector of P. Then exactly one of the following holds:*

(a) Axis plane*: w has one zero entry and two nonzero entries of opposite sign. Up to sign and a permutation of coordinates, $P = \{ax = by\}$ with $a, b \ge 1$ and $\gcd(a, b) = 1$.*

(b) Generic plane*: w has no zero entries, and exactly one entry has sign opposite to the other two. Up to sign and a permutation of coordinates, $P = \{ax + by = cz\}$ with $a, b, c \ge 1$ and $\gcd(a, b, c) = 1$.*

Each plane is represented exactly once if we record (a) by the position of the zero entry (3 choices) and the ordered pair (a, b), and (b) by the position of the odd-signed entry (3 choices) and the ordered triple (a, b, c).

Proof. Points of $\mathcal{B}_N$ have all coordinates positive. If all entries of w were ≥ 0, or all ≤ 0, then $w \cdot x \neq 0$ for every $x \in \mathcal{B}_N$. So w has entries of both signs. Two zero entries would force $w = \pm e_i$ and hence $x_i = 0$, which is impossible. This leaves cases (a) and (b). Replacing w by $-w$ fixes the sign convention, which gives uniqueness. □

3.4 Axis planes reduce to 2×2

Let $D(q, N)$ be the number of ordered parallel pairs in the rectangular grid $\{1, \ldots, q\} \times \{1, \ldots, N\}$:

$$D(q, N) = \#\{(k, z, k', z') : 1 \leq k, k' \leq q,\ 1 \leq z, z' \leq N,\ kz' = k'z\} = \sum_{\substack{u,v \geq 1 \\ \gcd(u,v)=1}} \min\Big(\big\lfloor \tfrac{q}{u} \big\rfloor, \big\lfloor \tfrac{N}{v} \big\rfloor\Big)^2.$$

In particular $D(N, N) = S_2(N)$.

Proposition 3.5. *Let $P = \{ax = by\}$ with $\gcd(a, b) = 1$, and put $q = \lfloor N/\max(a, b) \rfloor$. Then*

$$L_P = qN, \qquad I_P = q^2N^2 - D(q, N).$$

Proof. The solutions of $ax = by$ in positive integers are $(x, y) = k(b, a)$ with $k \geq 1$, and they lie in $\{1, \ldots, N\}^2$ exactly when $k \leq q$. The third coordinate is unconstrained. So $P \cap \mathcal{B}_N = \{(kb, ka, z) : 1 \leq k \leq q,\ 1 \leq z \leq N\}$, which is the image of the grid $\{1, \ldots, q\} \times \{1, \ldots, N\}$ under the injective linear map $(k, z) \mapsto (kb, ka, z)$. Linear maps preserve parallelism, so the parallel pairs in $P \cap \mathcal{B}_N$ correspond exactly to those in the grid. □

3.5 The decomposition

Combining Lemmas 3.1, 3.2 and 3.4 and Proposition 3.5, and writing $h(1) = 1$ and $h(m) = 2\varphi(m)$ for $m \geq 2$ (the number of coprime ordered pairs (a, b) with $\max(a, b) = m$), we obtain the following.

Theorem 3.6 (Exact decomposition)**.** *For every $N \geq 1$,*

$$S_3(N) \;=\; \underbrace{N^3 \sum_{m=1}^{N} g(m) \left\lfloor \tfrac{N}{m} \right\rfloor^2}_{\text{parallel pair}} \;+\; \underbrace{3 \sum_{m=1}^{N} h(m)\, q_m N (q_m^2 N^2 - D(q_m, N))}_{\text{axis planes}} \;+\; \underbrace{3 \sum_{\substack{a,b,c \geq 1 \\ \gcd(a,b,c)=1}} L_{abc}\, I_{abc}}_{\text{generic planes}},$$

where $q_m = \lfloor N/m \rfloor$, and L_{abc} and I_{abc} are the quantities L_P and I_P for the plane $ax + by = cz$. The last sum is finite: only planes with $a, b, c \leq N^2$ contribute.

We verified Theorem 3.6 against brute-force enumeration for all $N \leq 10$, computing the generic term directly. Figure 3 shows how the three terms share the total. The generic term grows from 10% of all singular matrices at $N = 2$ to 68% at $N = 34$ and 75% at $N = 100$, and it is the only term that can produce the factor $\log N$ in the asymptotics. Indeed, the first two terms are of exact order N^6, with explicit constants.

Proposition 3.7 (Asymptotics of the explicit terms)**.** *Let* $\kappa = \dfrac{2\zeta(2)}{\zeta(3)} - 1 = 1.7368655\ldots$. *Then*

(a) $A(N) = \kappa N^3 + O(N^2 \log N)$*, so the parallel-pair term is* $N^3 A(N) = \kappa N^6 + O(N^5 \log N)$*;*

(b) the axis-plane term is $3\kappa N^6 + O(N^5 \log N)$*.*

Consequently, the generic-plane term equals $S_3(N) - 4\kappa N^6 + O(N^5 \log N)$*.*

Proof. We use $G(x) = \#\{v \in \mathbb{Z}^3_{>0} \text{ primitive} : \|v\|_\infty \le x\}$. By Möbius inversion, $G(x) = \sum_{d \le x} \mu(d) \lfloor x/d \rfloor^3 = x^3/\zeta(3) + O(x^2)$ for $x \ge 1$, and $G(x) = 0$ for $x < 1$.

(a) An ordered parallel pair in $\mathcal{B}_N^2$ is $(k_1 v, k_2 v)$ with v primitive, $k_1, k_2 \ge 1$ and $\max(k_1, k_2) \cdot \|v\|_\infty \le N$. Exactly $2k-1$ pairs (k_1, k_2) have $\max(k_1, k_2) = k$, so

$$\begin{aligned} A(N) &= \sum_{k=1}^{N} (2k-1)\, G(N/k) \\ &= \frac{N^3}{\zeta(3)} \sum_{k \le N} \frac{2k-1}{k^3} + O\Big(N^2 \sum_{k \le N} \frac{2k-1}{k^2}\Big). \end{aligned}$$

The first sum is $2\zeta(2) - \zeta(3) + O(1/N)$ and the second is $O(\log N)$, which gives $A(N) = \kappa N^3 + O(N^2 \log N)$.

(b) Write $q_m = \lfloor N/m \rfloor$. By Theorem 3.6, the axis-plane term is

$$3N^3 \sum_{m \le N} h(m)\, q_m^3 \;-\; 3N \sum_{m \le N} h(m)\, q_m\, D(q_m, N).$$

Since $0 \le x^3 - \lfloor x \rfloor^3 \le 3x^2$ and $h(m) \le 2m$, the first sum is

$$\sum_{m \le N} h(m)\, q_m^3 = N^3 \sum_{m \le N} \frac{h(m)}{m^3} + O\Big(N^2 \sum_{m \le N} \frac{h(m)}{m^2}\Big).$$

Using $\sum \varphi(m) m^{-s} = \zeta(s-1)/\zeta(s)$, the series converges to

$$\sum_{m \ge 1} \frac{h(m)}{m^3} = 2 \sum_{m \ge 1} \frac{\varphi(m)}{m^3} - 1 = \frac{2\zeta(2)}{\zeta(3)} - 1 = \kappa,$$

and its tail beyond N is $O(1/N)$. The error sum is $O(\log N)$. Hence the first part is $3\kappa N^6 + O(N^5 \log N)$.

For the second part, $D(q, N) \le D(N, N) = S_2(N) = O(N^2 \log N)$ for $q \le N$ (Corollary 2.3), and $\sum_{m \le N} h(m) q_m \le \sum_{m \le N} 2m \cdot N/m = 2N^2$. So it is $O(N^5 \log N)$. The final statement follows by subtraction. □

The two explicit terms share the same constant κ, which is a coincidence of the two zeta-function evaluations rather than a structural fact that we know of. Table 3 shows the slow approach to the limits. The explicit terms contribute $4\kappa \approx 6.95$ to the coefficient of N^6. The generic planes contribute $C_3 N^6 \log N$ plus a second-order term $c_{\text{gen}} N^6$, with $c_{\text{gen}} \approx -9.686$ (Section 6.4), so the total second-order constant is $c_3 = 4\kappa + c_{\text{gen}} \approx -2.738$.

Remark 3.8 (Why the generic term resists a closed form)**.** For a single plane, $L_{abc} = \sum_{z=1}^{N} \#\{1 \le x, y \le N : ax + by = cz\}$. Each summand is a bounded version of the classical problem of

N	$A(N)/N^3$	axis-plane term$/N^6$
34	1.5897	4.7427
200	—	5.1086
600	—	5.1728
10^3	1.72696	—
10^4	1.73558	—
10^5	1.73669	—
limit	$\kappa = 1.73687$	$3\kappa = 5.21060$

Table 3: Convergence of the explicit terms of Theorem 3.6 to the limits of Proposition 3.7. $A(N)$ is computed from Lemma 3.2 with a sieve. The axis-plane term includes the $D(q, N)$ correction and is computed directly.

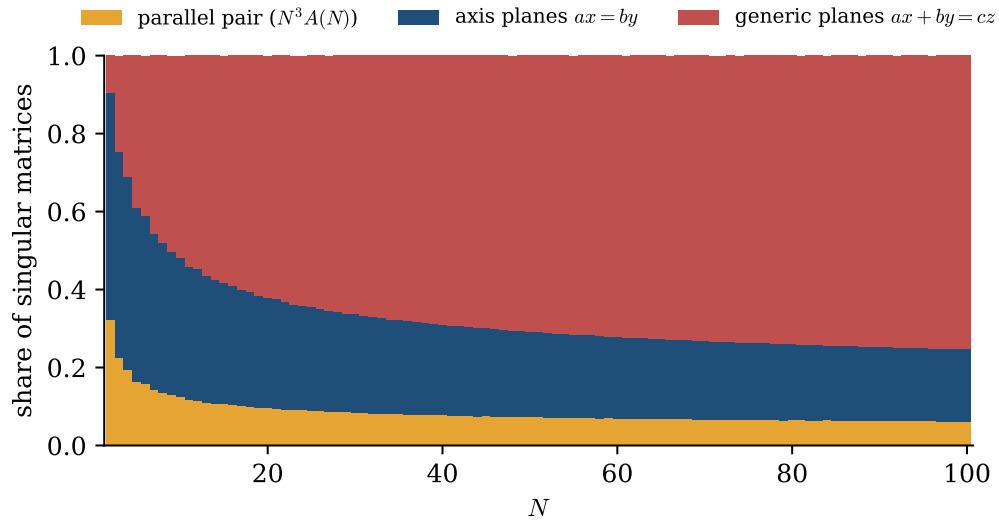


Figure 3: Share of the three terms of Theorem 3.6 in $S_3(N)$. The first two terms are computed from their closed forms, and the generic term is obtained by subtracting them from the exact count. Only the generic term grows like $N^6 \log N$, so its share tends to 1, but slowly.

counting solutions of $ax + by = n$. Popoviciu's theorem (see [4, §1.4]) gives the unbounded count $\frac{n}{ab} - \{\frac{b'n}{a}\} - \{\frac{a'n}{b}\} + 1$ when $\gcd(a, b) = 1$, where $aa' \equiv 1 \pmod b$ and $bb' \equiv 1 \pmod a$. With inclusion–exclusion for the bounds, this makes each L_{abc} an explicit sum of fractional parts of Dedekind-sum type. The obstruction is the I_{abc}-weighted sum over all (a, b, c), which mixes the arithmetic of infinitely many lattices. Rather than evaluate it exactly, we extract its leading behaviour.

4 The leading asymptotic constant

4.1 Plane sections of the cube

For a unit vector $u \in S^2$, let

$$\alpha(u) = \operatorname{area}(u^\perp \cap [0, 1]^3)$$

be the area of the section of the unit cube by the plane through the corner 0 with normal u. This is zero unless u has entries of both signs. Define

$$J = \int_{S^2} \alpha(u)^3 \, d\sigma(u),$$

where σ is surface measure on the sphere.

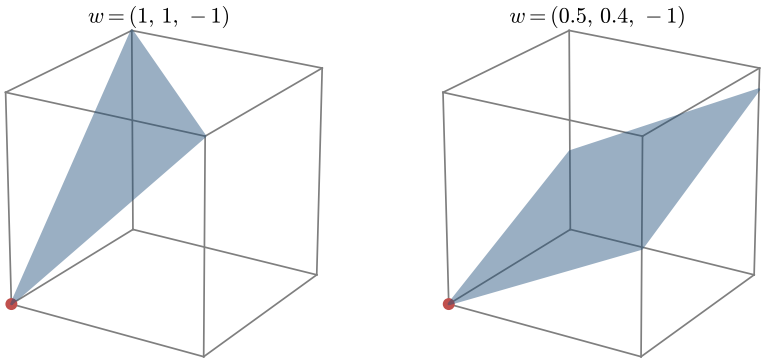


Figure 4: Two sections $w^\perp \cap [0,1]^3$ of the unit cube by planes through the corner 0 (red dot). Left: the plane $x + y = z$, whose section is a triangle. Right: the plane $0.5x + 0.4y = z$, whose section is a quadrilateral. The area α of such a section, scaled by N^2 and divided by $|w|$, approximates the number of rows of $\mathcal{B}_N$ lying in the plane.

4.2 Derivation of the main term

We follow Katznelson [13] and Wigman [25], and indicate where each step comes from.

Step 1. Reduce to planes. Every singular matrix has its rows in $w^\perp$ for some primitive $w \in \mathbb{Z}^3$, and $\pm w$ give the same plane. Rank-≤ 1 matrices, and matrices with two parallel rows, contribute only $O(N^6)$ (Lemmas 3.1 and 3.2; [25, §4]). Hence, to leading order,

$$S_3(N) \approx \tfrac{1}{2} \sum_{w \text{ primitive}} L_w^3, \qquad L_w = \#(w^\perp \cap \mathcal{B}_N).$$

Step 2. Count lattice points in each plane. The integer points of $w^\perp$ form a 2-dimensional lattice of covolume $|w|$ (Euclidean norm). The region $w^\perp \cap [0,N]^3$ is a convex polygon of area $N^2 \alpha(w/|w|)$. For typical w with $|w| \leq N^{2-\varepsilon}$,

$$L_w = \frac{N^2 \, \alpha(w/|w|)}{|w|}(1 + o(1)).$$

The exceptions are planes whose lattice is very elongated. For example, the axis plane $ax = by$ contains the short vector e_3, and when $N/2 < \max(a,b) \leq N$ it holds N rows although the formula predicts $N^2/\max(a,b)$, up to twice as many. Such planes form a small exceptional family. In Wigman's setting of primitive rows, controlling it requires Schmidt's equidistribution theorem [18, 25]. For our count, an elementary bound on the number of such planes suffices (Lemma 5.2).

Step 3. Sum over normal vectors. The summand $N^6\alpha(\hat{w})^3/\,|w|^3$ is homogeneous of degree -3 in w. Primitive vectors have density $1/\zeta(3)$, so

$$\tfrac{1}{2}\sum_{\substack{w \text{ primitive}\\ |w|\le cN^2}} \frac{N^6\,\alpha(\hat{w})^3}{|w|^3} = \frac{N^6}{2\zeta(3)}\int_{1\le|x|\le cN^2} \frac{\alpha(\hat{x})^3}{|x|^3}\,dx + O(N^6)$$

$$= \frac{N^6}{2\zeta(3)}\,J\,\log(N^2) + O(N^6).$$

The cutoff $|w| \lesssim N^2$ is where planes stop containing two independent rows, since a cross product $r_1 \times r_2$ has size $O(N^2)$. The constant c affects only the $O(N^6)$ term.

These three steps are made rigorous in Section 5, which proves the following.

Theorem 4.1 (Main theorem)**.**

$$S_3(N) = \frac{J}{\zeta(3)}\,N^6\log N + O(N^6), \qquad \textit{equivalently} \qquad \Pr_N[\det M = 0] = \frac{J}{\zeta(3)}\cdot\frac{\log N}{N^3} + O\Big(\frac{1}{N^3}\Big).$$

The error term $O(N^6)$ is the same as in Katznelson's theorem, which applies to bounded convex open sets containing the origin. The proof in Section 5 is elementary. It needs no equidistribution theorem, and the position of the origin at a corner of the cube plays no role.

4.3 Two consistency checks

The derivation above contains no free parameters, so it can be tested on cases where the answer is known.

Remark 4.2 (Recovering Shi's 2×2 constant)**.** In dimension 2, the same three steps give $S_2(N) \sim \frac{J_2}{2\zeta(2)}N^2\log N$, where $J_2 = \int_{S^1}\alpha_1(u)^2\,d\theta$ and $\alpha_1(u)$ is the length of the segment $u^\perp \cap [0,1]^2$. The cutoff is now $|w| \lesssim N$, giving $\log N$ rather than $2\log N$. A line through 0 with direction angle $\phi \in (0,\pi/2)$ meets the square in a segment of length $1/\max(\cos\phi, \sin\phi)$, and each such line has two unit normals, so

$$J_2 = 2\int_0^{\pi/2} \frac{d\phi}{\max(\cos\phi,\sin\phi)^2} = 4\int_0^{\pi/4} \sec^2\phi\,d\phi = 4.$$

Hence the constant is $\frac{4}{2\zeta(2)} = \frac{12}{\pi^2}$, matching Corollary 2.3 and Shi [20].

Remark 4.3 (Recovering Katznelson's constant for balls)**.** If the rows range over the Euclidean ball of radius T instead of a cube, every central section is a disc of area πT^2. The same computation gives $\frac{1}{\zeta(3)}\cdot 4\pi\cdot\pi^3\,T^6\log T = \frac{4\pi^4}{\zeta(3)}T^6\log T$. Katznelson's theorem, as quoted in [25], gives $\frac{(n-1)u_n}{\zeta(n)}T^{n^2-n}\log T$ with $u_3 = 2\pi^4$, which is the same constant.

4.4 Evaluating the integral

Lemma 4.4.

$$J = 6\int_0^\infty\int_0^\infty \alpha(a,b)^3\,da\,db, \qquad \alpha(a,b) := \text{area}\{(x,y)\in[0,1]^2 :\ ax+by\le 1\}.$$

Proof. Normals with entries of both signs fill six of the eight open octants, and by permuting coordinates and changing signs all six contribute equally. In the octant of $w = (a, b, -1)$ with $a, b > 0$, the section $w^{\perp} \cap [0,1]^3$ is the graph of $z = ax + by$ over the region $\{(x, y) \in [0,1]^2 : ax + by \le 1\}$. Its area is therefore $\alpha(a,b)\sqrt{1 + a^2 + b^2}$. Under the central projection $(a, b) \mapsto w/|w|$, surface measure is $d\sigma = (1 + a^2 + b^2)^{-3/2}\, da\, db$. The two factors $(1 + a^2 + b^2)^{\pm 3/2}$ cancel, leaving the integrand $\alpha(a, b)^3$. □

The function $\alpha(a, b)$ is piecewise elementary. The line $ax + by = 1$ cuts the unit square in one of four ways, giving the four regions of Fig. 5:

$$\alpha(a,b) = \begin{cases} 1, & \text{I: } a + b \le 1, \\ 1 - \dfrac{(a+b-1)^2}{2ab}, & \text{II: } a, b \le 1 < a + b, \\ \dfrac{1 - b/2}{a}, & \text{III: } b \le 1 \le a \quad (\text{III}'\text{: swap } a, b), \\ \dfrac{1}{2ab}, & \text{IV: } a, b \ge 1. \end{cases}$$

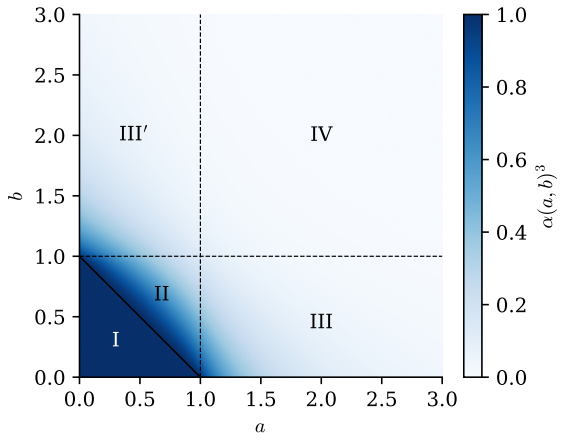


Figure 5: The integrand $\alpha(a, b)^3$ of Lemma 4.4 and its four regions. Regions I, III, III′ and IV contribute exactly 1 in total. Region II contributes $\frac{13}{2} - \frac{5\pi^2}{8}$, which is where π^2 enters the constant.

Lemma 4.5. *The regions I, III, III′ and IV contribute* $\frac{1}{2}$, $\frac{15}{64}$, $\frac{15}{64}$ *and* $\frac{1}{32}$ *respectively, for a total of* 1.

Proof. Region I is a triangle of area $\frac{1}{2}$ on which $\alpha = 1$. On region III, $\int_1^\infty a^{-3}\, da \int_0^1 (1 - \frac{b}{2})^3\, db = \frac{1}{2} \cdot \frac{15}{32}$. On region IV, $\int_1^\infty \int_1^\infty (2ab)^{-3}\, da\, db = \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2}$. □

Lemma 4.6.

$$P_2 := \iint_{II} \Big(1 - \frac{(a+b-1)^2}{2ab}\Big)^3\, da\, db = \frac{13}{2} - \frac{5\pi^2}{8}.$$

Proof. For fixed $a \in (0, 1)$, the integrand is a rational function of b, and integrating over $b \in [1 - a, 1]$ gives

$$\int_{1-a}^{1} \Big(1 - \frac{(a+b-1)^2}{2ab}\Big)^3\, db = R(a) + Q(a)\log(1-a),$$

with

$$Q(a) = \frac{3a^4 - 12a^3 + 30a^2 - 36a + 15}{8a^3}, \qquad R(a) = \frac{2a^5 - 12a^4 + 11a^3 + 68a^2 - 114a + 60}{32a^2}.$$

(This identity is mechanical to verify, and we also checked it numerically to 30 digits at several points.) Now expand

$$Q(a) = \tfrac{3a}{8} - \tfrac{3}{2} + \tfrac{15}{4a} - \tfrac{9}{2a^2} + \tfrac{15}{8a^3}, \qquad R(a) = \tfrac{2a^3-12a^2+11a+68}{32} + \tfrac{15}{8a^2} - \tfrac{57}{16a},$$

and integrate the pieces over $(0,1)$:

- $\int_0^1 \frac{3a}{8}\log(1-a)\,da = -\frac{3}{8}\cdot\frac{3}{4} = -\frac{9}{32}$ and $\int_0^1 -\frac{3}{2}\log(1-a)\,da = \frac{3}{2}$.
- $\int_0^1 \frac{15}{4a}\log(1-a)\,da = -\frac{15}{4}\cdot\frac{\pi^2}{6} = -\frac{5\pi^2}{8}$. This is the only source of π^2.
- The polynomial part of R integrates to $\frac{1}{32}(\frac{1}{2} - 4 + \frac{11}{2} + 68) = \frac{35}{16}$.
- The remaining singular terms are $G(a) = -\frac{9}{2a^2}\log(1-a) + \frac{15}{8a^3}\log(1-a) + \frac{15}{8a^2} - \frac{57}{16a}$. Substituting $\log(1-a) = -\sum_{k\ge1} a^k/k$, the coefficients of a^{-2} and a^{-1} cancel exactly, leaving
$$\int_0^1 G = \frac{9}{2}\sum_{k\ge2}\frac{1}{k(k-1)} - \frac{15}{8}\sum_{k\ge3}\frac{1}{k(k-2)} = \frac{9}{2} - \frac{15}{8}\cdot\frac{3}{4} = \frac{99}{32}.$$

The total is $-\frac{9}{32} + \frac{48}{32} + \frac{70}{32} + \frac{99}{32} - \frac{5\pi^2}{8} = \frac{13}{2} - \frac{5\pi^2}{8}$. Numerical quadrature at 60-digit precision agrees. □

Theorem 4.7. *$J = 45 - \dfrac{15\pi^2}{4} = 7.98898349591\ldots$, and therefore the constant in Theorem 4.1 is*

$$C_3 = \frac{J}{\zeta(3)} = \frac{15\,(12-\pi^2)}{4\,\zeta(3)} = 6.6460942696772\ldots$$

Proof. By Lemmas 4.4 to 4.6, $J = 6(1 + \frac{13}{2} - \frac{5\pi^2}{8}) = 45 - \frac{15\pi^2}{4}$. □

Remark 4.8 (Interpretation)**.** For a random 3×3 matrix with entries uniform on $\{1,\ldots,N\}$,

$$\Pr[\det M = 0] \approx 6.646\,\frac{\log N}{N^3}.$$

For $N = 100$ this gives about 3×10^{-5}, and for $N = 10^6$ about 9×10^{-17}. By comparison, the probability that two rows are *equal* is only about $3/N^3$. The extra factor $\log N$ comes from the many planes $ax + by = cz$ with moderately large coefficients, each of which contains its own small family of rows.

4.5 The constant as a determinant density

The integral J has a simple probabilistic meaning. For $n \ge 2$ let

$$J_n = \int_{S^{n-1}} \mathrm{vol}_{n-1}\big(u^\perp \cap [0,1]^n\big)^n\, d\sigma(u),$$

so that $J_3 = J$, and let p_n be the probability density of $\det M$, where M is an $n\times n$ random matrix whose entries are independent and uniform on $[0,1]$.

Proposition 4.9 (Determinant density)**.** *For every $n \geq 2$, $J_n = 2\,p_n(0)$.*

Proof. Let $K = [0,1]^n$, and let the rows of M be $r_1, \dots, r_n$, independent and uniform in K. Expanding along the first row, $\det M = r_1 \cdot C$, where $C = C(r_2, \dots, r_n)$ is the vector of cofactors. It is orthogonal to $r_2, \dots, r_n$, and $|C|$ is the $(n-1)$-volume of the parallelotope they span. Given $r_2, \dots, r_n$ with $C \neq 0$, the linear form $r_1 \cdot C$ has density $\mathrm{vol}_{n-1}(C^\perp \cap K)/\,|C|$ at 0. Hence

$$p_n(0) = \int_{K^{n-1}} \frac{\mathrm{vol}_{n-1}(\hat{C}^\perp \cap K)}{|C|}\, dr_2 \cdots dr_n.$$

The linear Blaschke–Petkantschin formula [19, Thm. 7.2.1], for $n-1$ points in $\mathbb{R}^n$, states that

$$\int_{(\mathbb{R}^n)^{n-1}} f\, dx_2 \cdots dx_n = \frac{1}{2} \int_{S^{n-1}} \int_{(u^\perp)^{n-1}} f \cdot |C|\; d\lambda_{u^\perp}^{n-1}\, d\sigma(u),$$

where $\lambda_{u^\perp}$ is Lebesgue measure on the hyperplane $u^\perp$. Apply it to the integrand above. The factor $|C|$ cancels, $\hat{C} = \pm u$, and each of the $n-1$ inner integrals contributes $\mathrm{vol}_{n-1}(u^\perp \cap K)$. So $p_n(0) = \frac{1}{2} \int_{S^{n-1}} \mathrm{vol}_{n-1}(u^\perp \cap K)^n\, d\sigma(u) = J_n/2$. □

For $n = 2$ this can be checked by hand: $\det M = ad - bc$, the products ad and bc have density $-\log t$ on $(0,1)$, so $p_2(0) = \int_0^1 \log^2 t\, dt = 2 = J_2/2$. A Monte Carlo check with 8×10^7 samples, extrapolating $\Pr[|\det M| < \varepsilon]/2\varepsilon$ to $\varepsilon \to 0$, gives 1.9986, 3.9934 and 7.140 for $n = 2, 3, 4$, against 2, 3.99449 and 7.13037.

Corollary 4.10. *The density at 0 of the determinant of a random 3×3 matrix with independent uniform $[0,1]$ entries is*

$$p_3(0) = \frac{45}{2} - \frac{15\pi^2}{8} = 3.99449\ldots,$$

and the leading constants of Corollary 2.3, Theorem 4.1 and (2) below can be written as

$$C_n = \frac{(n-1)\,p_n(0)}{\zeta(n)}.$$

So the leading constant is a product of an arithmetic factor $(n-1)/\zeta(n)$ and the purely probabilistic quantity $p_n(0)$. The proof uses only that the rows are independent and uniform in a region K. For rows uniform in the unit ball, it identifies Wigman's constant u_n [25], via $J_n = 2u_n$ (Remark 4.3), with $\mathrm{vol}(B)^n$ times the density at 0 of the determinant of a matrix with independent rows uniform in the ball.

4.6 Beyond 3×3: the 4×4 constant

The three steps of the derivation work in any dimension. In $\mathbb{R}^n$, the integer points of $w^\perp$ form a lattice of covolume $|w|$, a box hyperplane holds about $N^{n-1}\,\mathrm{vol}_{n-1}\,/\,|w|$ rows, and the sum over $|w| \lesssim N^{n-1}$ produces $\log(N^{n-1}) = (n-1)\log N$. This leads to the prediction

$$S_n(N) \sim C_n\, N^{n^2-n} \log N, \qquad C_n = \frac{(n-1)\,J_n}{2\,\zeta(n)}, \qquad J_n = \int_{S^{n-1}} \mathrm{vol}_{n-1}(u^\perp \cap [0,1]^n)^n\, d\sigma(u). \tag{2}$$

For $n = 2, 3$ this gives $12/\pi^2$ and C_3. Replacing the cube by a ball, it gives Katznelson's constant $(n-1)u_n/\zeta(n)$ for every n: every central section is then a ball of volume v_{n-1}, so

$J_n = v_{n-1}^n \, |S^{n-1}| = 2u_n$ in Wigman's notation, as in Remark 4.3. For $n \geq 4$, Wigman notes that the elongated-lattice difficulty of the case $n = 3$ does not arise [25, §1.3], and the proof in Section 5 should extend, with hyperplanes in place of planes. We have not carried this out, so for $n = 4$ the formula remains a prediction.

Lemma 4.11 (Chart identity in dimension n)**.** *Let $u \in S^{n-1}$ lie in the open orthant containing $w = (a_1, \dots, a_{n-1}, -1)$ with all $a_i > 0$. Then*

$$\mathrm{vol}_{n-1}(u^\perp \cap [0,1]^n)^n \, d\sigma(u) = \alpha_{n-1}(\boldsymbol{a})^n \, d^{n-1}\boldsymbol{a}, \qquad \alpha_{n-1}(\boldsymbol{a}) = \mathrm{vol}_{n-1}\{x \in [0,1]^{n-1} : \ \boldsymbol{a} \cdot x \leq 1\}.$$

The same holds for any other mixed-sign orthant, with the corresponding region $\{x \in [0,1]^{n-1} : 0 \leq \boldsymbol{a} \cdot_\pm x \leq 1\}$, where $\cdot_\pm$ carries the signs of the orthant.

Proof. As in Lemma 4.4: the section is the graph of $x_n = \boldsymbol{a} \cdot_\pm x$ over that region, so its volume is the region's volume times $(1 + \|\boldsymbol{a}\|^2)^{1/2}$. Under central projection, $d\sigma = (1 + \|\boldsymbol{a}\|^2)^{-n/2} \, d^{n-1}\boldsymbol{a}$. Raising the volume to the power n, the two factors cancel exactly. □

For $n = 4$, the 14 mixed-sign orthants of $\mathbb{R}^4$ fall into two classes under coordinate permutations and $u \mapsto -u$. Eight orthants have three entries of one sign and one of the other, and their sections are graphs over $\{ax + by + cz \leq 1\}$. Six orthants have two entries of each sign, and their sections are graphs over the slab $\{0 \leq ax + by - cz \leq 1\}$. Hence

$$J_4 = 8 \int_{\mathbb{R}^3_{>0}} \alpha_3(a,b,c)^4 \, da\, db\, dc \ + \ 6 \int_{\mathbb{R}^3_{>0}} \beta_3(a,b,c)^4 \, da\, db\, dc,$$

with $\alpha_3(a,b,c) = F_{a,b,c}(1)$ and $\beta_3(a,b,c) = F_{a,b,c}(1+c) - F_{a,b,c}(c)$, where

$$F_{a,b,c}(t) = \mathrm{vol}\{x \in [0,1]^3 : \ ax + by + cz \leq t\} = \frac{1}{6abc} \sum_{S \subseteq \{a,b,c\}} (-1)^{|S|} \Big(t - \sum_{s \in S} s\Big)_+^3.$$

The formula for β_3 follows by substituting $z \mapsto 1 - z$, which turns $0 \leq ax + by - cz \leq 1$ into $c \leq ax + by + cz \leq 1 + c$. The formula for F is the standard inclusion–exclusion over the vertices of the cube. Both integrals converge, since α_3 and β_3 are at most $\min(1, 1/\max(a,b,c))$: in each region, every coordinate direction meets the region in an interval of length at most $1/a$, $1/b$ or $1/c$ respectively.

Exact inner integrations. Write $I_\alpha = \int_{\mathbb{R}^3_{>0}} \alpha_3^4$ and $I_\beta = \int_{\mathbb{R}^3_{>0}} \beta_3^4$, so that $J_4 = 8I_\alpha + 6I_\beta$. Two of the three integrations can be done exactly, as follows.

(i) *The c-integral.* For fixed (a,b), α_3 and β_3 are piecewise of the form (cubic polynomial in c)$/(6abc)$, with breakpoints at $c \in \{1-a, 1-b, 1-a-b, 1\}$ for α_3 and $c \in \{a, b, a+b, a-1, b-1, a+b-1\}$ for β_3. So $H(a,b) = \int_0^\infty (\cdot)^4 \, dc$ is an explicit sum of rational functions and rational multiples of logarithms of these breakpoints. On each region of the (a,b)-plane where the breakpoints are ordered, the denominators turn out to be monomials $a^i b^j$.

(ii) *The b-integral.* Hence only the antiderivatives $\int b^m \, db$ and $\int b^m \log(b+s) \, db$ with $m \in \mathbb{Z}$ and $s \in \{0, a, a-1, -1\}$ are needed. They are elementary, except that $m = -1$ produces a dilogarithm. We verified each antiderivative by differentiation.

(iii) *The a-integral.* What remains is a one-dimensional integral of rational functions, logarithms and dilogarithms in a. We evaluated it by tanh–sinh quadrature at 110 to 150 digits, raising the working precision near the endpoints to absorb cancellation.

For I_β the unbounded regions can be avoided. Splitting the $(2,2)$-orthant according to which coordinate of the normal is largest, and using in each piece the chart that divides by that coordinate, gives $I_\beta = 4\int_{[0,1]^3} \beta_3^4$, where the factor 4 counts the four cones. This leaves only two bounded (a,b)-regions.

Numerical identification 4.12 (Closed form of J_4)**.** *To the precision of the computation,*

$$I_\alpha = \frac{2200223}{38880} + \frac{1351}{648}\pi^2 - \frac{116389}{216}\zeta(3) + \frac{333724}{405}\log 2 = 0.61323405860879703879\ldots,$$
$$I_\beta = \frac{7271461}{87480} - \frac{4789}{1458}\pi^2 + \frac{4985}{54}\zeta(3) - \frac{31184}{135}\log 2 = 1.55914399309801657934\ldots,$$

and therefore

$$J_4 = \frac{1387213}{1458} - \frac{736}{243}\pi^2 - \frac{101434}{27}\zeta(3) + \frac{421696}{81}\log 2 = 14.2607364274584757864\ldots$$

The two relations were found by the PSLQ algorithm [8] with the basis 1, π^2, $\zeta(3)$, $\log 2$, $\log^2 2$, $\log^3 2$, $\pi^2 \log 2$, the constants that iterated integrals of this kind can produce up to weight three. PSLQ recovers them from 80 and 85 digits respectively. From 60 digits or fewer it finds nothing, because coefficients of this size cannot be detected at that precision. With the full precision the relations hold to 142 and 101 digits, far beyond what a chance relation with coefficients of this size could achieve. The values also agree with an independent double-double tanh–sinh computation on the simplex, to 27 and 21 digits, and with a Gauss–Legendre computation in the (a,b,c) variables, to its 5 digits. These closed forms are therefore identified, not proved. Since the first two integrations are exact, carrying out the last one-dimensional integral symbolically would turn it into a theorem.

Prediction 4.13 (The 4×4 constant)**.** *Assuming* (2) *holds for* $n = 4$*,*

$$S_4(N) \sim C_4\, N^{12}\log N, \qquad C_4 = \frac{3J_4}{2\zeta(4)} = \frac{135\,J_4}{\pi^4} = 19.76406305890759\ldots,$$

that is,

$$C_4 = \frac{1}{\pi^4}\Big(\frac{6936065}{54} - \frac{3680}{9}\pi^2 - 507170\,\zeta(3) + \frac{2108480}{3}\log 2\Big),$$

so a random 4×4 *matrix with entries uniform on* $\{1,\ldots,N\}$ *is singular with probability about* $19.76\,\log N/N^4$*. Equivalently, by Proposition 4.9, the density at* 0 *of the determinant of a random* 4×4 *matrix with uniform* $[0,1]$ *entries is* $p_4(0) = J_4/2 = 7.1303682137\ldots$

n	$J_n = 2p_n(0)$	$C_n = \frac{(n-1)J_n}{2\zeta(n)}$	Status
2	4	$12/\pi^2 \approx 1.2159$	proved (Corollary 2.3; Shi [20])
3	$45 - 15\pi^2/4$	$\frac{15(12-\pi^2)}{4\zeta(3)} \approx 6.6461$	proved (Theorem 4.1)
4	$14.26073642\ldots$	$19.76406305\ldots$	predicted (Section 4.6)

Table 4: Leading constants C_n in $S_n(N) \sim C_n N^{n^2-n}\log N$ for entries in $\{1,\ldots,N\}$. For $n = 4$, the closed form of J_4 is not proved (Numerical identification 4.12), and C_4 is a prediction (Prediction 4.13).

Remark 4.14 (Credit)**.** The reduction of J_4 to the two integrals above was proposed by Gemini (Google), an AI model, in response to a draft of this note. We checked the chart identity and the orthant count, and carried out the evaluation.

5 Proof of the main theorem

We now prove Theorem 4.1. The proof follows the three steps of Section 4, but replaces each approximation by an explicit bound. Implied constants are absolute.

Notation. For a primitive $w \in \mathbb{Z}^3$, let $\Lambda_w = w^\perp \cap \mathbb{Z}^3$. This is a rank-2 lattice of determinant $|w|$, and we write $\mu_1(w) \le \mu_2(w)$ for its successive minima. Let $K_w = w^\perp \cap [0,N]^3$, a convex polygon of diameter at most $\sqrt{3}N$ and area $N^2\alpha(\hat{w})$, and let

$$M_w = \frac{\operatorname{area}(K_w)}{|w|} = \frac{N^2\alpha(\hat{w})}{|w|}.$$

Each plane P through 0 is $w^\perp$ for exactly two primitive vectors $\pm w$. Call P *admissible* if $P \cap \mathcal{B}_N$ contains two linearly independent vectors. Recall $L_P = \#(P \cap \mathcal{B}_N)$ and $I_P = L_P^2 - \mathrm{par}_P$, where par_P is the number of ordered parallel pairs in $P \cap \mathcal{B}_N$.

5.1 Four lemmas

Lemma 5.1 (Lattice points in planar convex sets)**.** *Let Λ be a rank-2 lattice in a plane, with determinant d and first minimum μ_1, and let K be a compact convex subset of that plane of diameter D. Then*

$$\left|\#(\Lambda \cap K) - \operatorname{area}(K)/d\right| \le \Big(2 + \frac{2}{\sqrt{3}}\Big)\frac{D}{\mu_1} + 1.$$

Proof. Let v_1 be a shortest nonzero vector. The lattice lies on parallel lines ℓ_j in direction v_1, spaced $h = d/\mu_1$ apart, and Hermite's inequality $\mu_1^2 \le \frac{2}{\sqrt{3}}\, d$ gives $h \ge \frac{\sqrt{3}}{2}\mu_1$. At most $D/h + 1$ of the lines meet K. Each meets K in a segment of some length c_j, which contains $c_j/\mu_1 + \theta_j$ lattice points with $|\theta_j| \le 1$.

Let $f(s)$ be the length of the chord of K at transverse coordinate s. It is concave on its support and $0 \le f \le D$, so it is nondecreasing up to a maximum point and nonincreasing after it. On each of the two monotone pieces, compare the value at each sample point with the integral of f over the adjacent interval of length h, on one side for an upper bound and on the other side for a lower bound. Only the sample point nearest the maximum has no such interval, so the sum and the integral differ by at most hD on each piece. Hence $\left|h\sum_j c_j - \operatorname{area}(K)\right| \le 2hD$, and

$$\Big|\#(\Lambda \cap K) - \frac{\operatorname{area}(K)}{h\mu_1}\Big| \le \frac{2D}{\mu_1} + \frac{D}{h} + 1 \le \Big(2 + \frac{2}{\sqrt{3}}\Big)\frac{D}{\mu_1} + 1.$$

Finally, $h\mu_1 = d$. □

No assumption on the position of the origin, or on K containing lattice points, is needed. This is why the corner of the cube causes no difficulty.

Lemma 5.2 (Few planes with short vectors)**.** *For $R \ge 1$ and $1 \le t \le R$,*

$$\#\{w \in \mathbb{Z}^3 \text{ primitive} : \ |w| \le R,\ \mu_1(w) \le t\} \ll R^2t^2.$$

Moreover $\mu_1(w) \le (4/3)^{1/4}\,|w|^{1/2}$ for every primitive w.

Proof. If $\mu_1(w) \le t$, then w is orthogonal to some primitive v with $|v| \le t$, so $w \in \Lambda_v$. By Lemma 5.1 applied to a disc of radius R in $v^\perp$, the number of such w for a given v is at most $\pi R^2/|v| + O(R+1)$. Summing over primitive v with $|v| \le t$ gives $\ll R^2 \sum_{|v|\le t} |v|^{-1} + Rt^3 \ll R^2t^2 + Rt^3 \ll R^2t^2$, since $t \le R$. The second statement is Hermite's inequality $\mu_1^2 \le \frac{2}{\sqrt{3}} d$. □

Lemma 5.3 (Regularity of the section area)**.** *The function α is bounded on S^2. Let $E = \{\pm e_1, \pm e_2, \pm e_3\}$. Then $|\alpha(u) - \alpha(v)| \ll |u - v|/\delta$ whenever the shorter great-circle arc from u to v stays at distance at least δ from E.*

Proof. The section lies in the ball of radius $\sqrt{3}$, so $\alpha \le 3\pi$. Given u, choose k with $|u_k| = \max_i |u_i| \ge 1/\sqrt{3}$, and let i, j be the other two indices. The section $u^\perp \cap [0,1]^3$ is the graph of $x_k = ax_i + bx_j$, where $(a,b) = -(u_i, u_j)/u_k \in [-1,1]^2$, over the region $\Omega(a,b) = \{(x_i, x_j) \in [0,1]^2 : 0 \le ax_i + bx_j \le 1\}$. Hence $\alpha(u) = \operatorname{area}\Omega(a,b)/|u_k|$.

The boundary of Ω inside the square consists of at most two segments of the lines $ax + by = 0$ and $ax + by = 1$, each of length at most $\sqrt{2}$. Changing (a,b) by η moves each point p of these lines by at most $|p|\,|\eta|/\sqrt{a^2+b^2} \le \sqrt{2}\,|\eta|/\sqrt{a^2+b^2}$ in the normal direction. So $\operatorname{area}\Omega$ is continuous away from $(a,b) = 0$, and wherever it is differentiable its gradient is at most $2 \cdot \sqrt{2} \cdot \sqrt{2}/\sqrt{a^2+b^2} = 4/\sqrt{a^2+b^2}$. Moreover, $\operatorname{area}\Omega$ is given by finitely many rational expressions in (a,b), which change only when one of the two lines passes through a corner of the square, that is, on the lines $a = 0$, $b = 0$, $a + b = 0$, $a = 1$, $b = 1$ and $a + b = 1$. Finally, $\sqrt{a^2+b^2} \ge (u_i^2 + u_j^2)^{1/2} \ge \operatorname{dist}(u, \{\pm e_k\})/\sqrt{2}$, and $u \mapsto (a,b)$ has bounded derivatives where $|u_k| \ge 1/\sqrt{3}$. Hence $|\nabla\alpha(u)| \ll 1/\operatorname{dist}(u,E)$ wherever α is differentiable. In the variable u, each of the six lines above is a great circle (for example, $a + b = 1$ becomes $u_i + u_j + u_k = 0$). So an arc that avoids E meets only finitely many pieces, α is continuous along it, and integrating the gradient bound along the arc gives the claim. □

At the six points of E, α is genuinely discontinuous: the plane $u^\perp$ then contains a face of the cube.

Lemma 5.4 (Weighted sum over primitive vectors)**.** *Let $\varphi : S^2 \to \mathbb{R}$ be bounded and satisfy the conclusion of Lemma 5.3. Then, for $X \ge 2$,*

$$\sum_{\substack{w \text{ primitive} \\ |w| \le X}} \frac{\varphi(\hat{w})}{|w|^3} = \frac{1}{\zeta(3)} \Big(\int_{S^2} \varphi \, d\sigma \Big) \log X + O(1).$$

Proof. First sum over all nonzero $w \in \mathbb{Z}^3$, one dyadic shell $X' < |w| \le 2X'$ at a time, and compare with the integral of $\varphi(\hat{x})\,|x|^{-3}$ over the shell using the unit cubes Q_w centred at the lattice points.

The $O(X'^2)$ cubes meeting the boundary of the shell contribute $O(X'^{-1})$ to the difference. For the other cubes and $x \in Q_w$, we have $|\hat{x} - \hat{w}| \ll 1/X'$ and $\big||x|^{-3} - |w|^{-3}\big| \ll X'^{-4}$. Put $\delta = X'^{-1/3}$. We may assume that X' is large enough for $|\hat{x} - \hat{w}| \le \delta$, since the finitely many smaller shells contribute $O(1)$. If $\operatorname{dist}(\hat{w}, E) \ge 2\delta$, the arc from $\hat{w}$ to $\hat{x}$ then stays at distance at least δ from E, so $|\varphi(\hat{x}) - \varphi(\hat{w})| \ll 1/(\delta X')$. These $O(X'^3)$ points contribute $O(1/(\delta X') + 1/X')$ in total. The remaining points lie in six cones of angular radius 2δ, which contain $O(X'^3\delta^2 + X'^2)$ lattice points, each contributing $O(X'^{-3})$.

So each shell contributes $O(X'^{-2/3})$ to the difference, and summing over shells gives

$$\sum_{1 \le |w| \le X} \varphi(\hat{w})\,|w|^{-3} = \Big(\int_{S^2} \varphi \Big) \log X + O(1).$$

Möbius inversion over the multiples of each primitive vector then gives the primitive sum $\sum_{d\le X}\mu(d)d^{-3}\big[(\int\varphi)\log(X/d)+O(1)\big]$, which equals the stated main term plus $O(1)$, because $\sum_d d^{-3}\log d<\infty$. □

5.2 Proof of Theorem 4.1

Step 1: an exact identity. Non-admissible planes have $I_P=0$. So Lemma 3.1 gives

$$S_3(N)=N^3A(N)+\sum_{P\text{ adm}}L_P^3-\sum_{P\text{ adm}}L_P\,\mathrm{par}_P, \tag{3}$$

and $N^3A(N)=O(N^6)$ by Proposition 3.7.

Step 2: admissible planes. Let $P=w^\perp$ be admissible. Its two independent box vectors have length at most $\sqrt{3}N$, so $\mu_2(w)\le\sqrt{3}N$. Since $|w|=\det\Lambda_w\le\mu_1\mu_2$, it follows that

$$|w|\le 3N^2\qquad\text{and}\qquad \mu_1(w)\ge|w|/(\sqrt{3}N).$$

Lattice points of K_w outside $\mathcal{B}_N$ have a zero coordinate. So they lie on the three lines $w^\perp\cap\{x_i=0\}$ (which are lines, since $w\neq\pm e_i$). Each such line holds at most $\sqrt{3}N/\mu_1+1$ points of K_w. With Lemma 5.1, this gives

$$L_P=M_w+E_w,\qquad |E_w|\ll N/\mu_1(w),\qquad M_w\ll N^2/|w|,$$

where the constant terms have been absorbed using $\mu_1\le\mu_2\le\sqrt{3}N$.

Step 3: parallel pairs. Each point of $P\cap\mathcal{B}_N$ lies on a unique line through 0. On such a line, the box points are the multiples ku, $1\le k\le\sqrt{3}N/|u|$, of a primitive $u\in\Lambda_w$, and $|u|\ge\mu_1$. Hence $\mathrm{par}_P\le L_P\cdot\sqrt{3}N/\mu_1$, and so $L_P\,\mathrm{par}_P\ll(M_w^2+E_w^2)N/\mu_1$.

Step 4: error sums. Since $L^3-M^3\ll M^2|E|+|E|^3$, Steps 2 and 3 turn (3) into

$$S_3(N)=\sum_{P\text{ adm}}M_w^3+O\Big(N^6+\sum_{P\text{ adm}}\Big(\frac{N^5}{|w|^2\,\mu_1}+\frac{N^3}{\mu_1^3}\Big)\Big).$$

Split the admissible w into dyadic shells $R<|w|\le 2R$ with $R\le 3N^2$. In a shell, μ_1 ranges over $[t_0,t_1]$, where $t_0=\max(1,R/(\sqrt{3}N))$ and $t_1=(4/3)^{1/4}(2R)^{1/2}$. Let $F(t)$ count the w in the shell with $\mu_1\le t$. Lemma 5.2, applied with $2R$ in place of R, gives $F(t)\ll R^2t^2$ for $1\le t\le t_1$ (note that $t_1\le 2R$), and in particular $F(t_1)\ll R^3$. For a decreasing g, partial summation gives

$$\sum_w g(\mu_1)\le g(t_1)F(t_1)+\int_{t_0}^{t_1}F(t)\,|g'(t)|\,dt.$$

With $g=t^{-1}$ this is $\ll R^{5/2}$, and with $g=t^{-3}$ it is $\ll R^{3/2}+R^2/t_0$. So each shell contributes $\ll N^5R^{1/2}+N^3(R^{3/2}+\min(R^2,\sqrt{3}NR))$. Summing over dyadic $R\le 3N^2$ gives $O(N^6)$.

Step 5: non-admissible planes. We have $\sum_{P\text{ adm}}M_w^3=\frac12\sum_{|w|\le 3N^2}M_w^3-\sum'M_w^3$, where the first sum is over primitive w and $\sum'$ is over non-admissible planes with $|w|\le 3N^2$.

The three coordinate planes contribute $3N^6$. For any other non-admissible plane, the lattice points of K_w lie on at most four lines through 0 (one containing all points of $P\cap\mathcal{B}_N$, and the three lines $P\cap\{x_i=0\}$). So $\#(\Lambda_w\cap K_w)\ll N/\mu_1+1$, and by Lemma 5.1 also $M_w\ll N/\mu_1+1$. Together with $M_w\ll N^2/|w|$, each shell contributes

$$\ll R^3+\Big(\frac{N^2}{R}\Big)^3F\Big(\frac{R}{N}\Big)+N^3\sum_{\mu_1>R/N}\mu_1^{-3}\ll R^3+N^4R+N^3\Big(R^{3/2}+\frac{R^2}{\max(1,R/N)}\Big).$$

Here R^3 bounds the contribution of the constant term, the second term covers the w with $\mu_1 \le R/N$ (there are none unless $R \ge N$), and the third covers $\mu_1 > R/N$, where $\mu_1 \ge \max(1, R/N)$ and the partial summation of Step 4 applies. This again sums to $O(N^6)$ over $R \le 3N^2$.

Step 6: the main term. Lemma 5.3 shows that $\varphi = \alpha^3$ satisfies the hypothesis of Lemma 5.4. With $X = 3N^2$, that lemma gives

$$\frac{1}{2}\sum_{|w|\le 3N^2} M_w^3 = \frac{N^6}{2}\Big(\frac{J}{\zeta(3)}\log(3N^2) + O(1)\Big) = \frac{J}{\zeta(3)}N^6 \log N + O(N^6).$$

Combining Steps 1–6 proves the theorem. □

Remark 5.5 (Comparison with Katznelson and Wigman)**.** The planes $x - y + cz = 0$ with $1 \le c \le N-2$ show that the error in Step 2 really depends on μ_1 and not only on $|w|$. Each is admissible and contains $(1,1,0)$, so $\mu_1 = \sqrt{2}$. A direct count gives $M_w = N^2/(2c)$ and

$$E_w = -\frac{N}{2} + \frac{r(c-r)}{2c}, \qquad r = N - c\Big\lfloor \frac{N-1}{c} \Big\rfloor \in [1, c],$$

so $|E_w| > 3N/8$ for every c, although $N/|w|^{1/2} \approx N/\sqrt{c}$. Lemma 5.2 shows that planes with a short vector are rare enough for their total contribution to be $O(N^6)$.

In Wigman's count of matrices with primitive rows, the analogue of Lemma 5.1 carries an extra factor $\log T$ coming from Möbius inversion. That is what forces him to use Schmidt's equidistribution theorem, and what weakens his error term to $O(T^6 \log\log T)$. For all integer rows no such loss occurs, consistent with Katznelson's $O(T^6)$.

As a numerical check, we enumerated every admissible plane for $N = 12$, 16, 20 and 24, with its exact point count and successive minima. The identity of Step 1 then reproduces $S_3(N)$ exactly. In all four cases $|E_w| \le 1.12\,(N/\mu_1 + 1)$ and $\mathrm{par}_P \le 0.93\, L_P \sqrt{3} N/\mu_1$. The two error sums of Step 4, divided by N^6, change little between $N = 12$ and $N = 24$ (from 4.66 to 5.37, and from 1.87 to 1.97), and $\sum_{\mathrm{adm}} L_P \,\mathrm{par}_P$ stays between $2.57N^6$ and $2.65N^6$. In Lemma 5.2 the ratio of the count to R^2t^2 stays below 6.92 for $R \le 300$. The programs are listed in Appendix B.

6 Numerical evidence

6.1 Exact counting algorithms

Brute force over N^9 matrices is hopeless beyond $N \approx 8$. Our first algorithm uses

$$S_3(N) = \sum_{(r_1,r_2)\in\mathcal{B}_N^2} L(r_1 \times r_2), \qquad L(c) = \#\{x \in \mathcal{B}_N : \; c \cdot x = 0\},$$

with $L(0) = N^3$. For $c = (p,q,r) \ne 0$ with $r \ne 0$, and each $x \in \{1,\dots,N\}$, the pairs (y,z) with $qy + rz = -px$ lie on a single residue class of y modulo $r/\gcd(q,r)$, intersected with an explicit interval. So $L(c)$ costs $O(N)$, and the total cost is $O(N^7)$. Its output agrees with brute force for $N \le 8$.

A faster algorithm works line by line. Group the first row r_1 by the line ℓ through 0 that contains it, and let k_ℓ be the number of box points on ℓ. Every other box point y determines the plane $\mathrm{span}(\ell, y)$. Writing m_P for the number of box points of a plane $P \supset \ell$ that are not on ℓ, one checks that

$$S_3(N) = \sum_{\ell} k_\ell\Big(2k_\ell N^3 - k_\ell^2 + \sum_{P\supset\ell} m_P^2\Big).$$

For each line, the planes through it are found by hashing the reduced direction of the image of y in the two-dimensional quotient lattice $\mathbb{Z}^3/\mathbb{Z}v$, where v spans ℓ. Processing the points y in shells of increasing $\max(y)$ gives $\sum_P m_P^2$ for all box sizes at once. Using the symmetry under permutations of coordinates, one pass costs $O(N_{\max}^6)$ and returns $S_3(N)$ for every $N \le N_{\max}$. With $N_{\max} = 100$ it took about 15 minutes on four cores. It reproduces all values from the first algorithm. The values for $N \le 100$ are listed in Appendix A.

6.2 Comparison with the main theorem

Figure 6(a) plots $S_3(N)/N^6$ against $\log N$. By Theorem 4.1, the points stay within bounded distance of a line of slope C_3, and they do. The residual $c_3(N) := S_3(N)/N^6 - C_3 \log N$ in Fig. 6(b) is still rising slowly at $N = 100$, where it equals -3.28. Figure 7 explains why. The explicit terms of Proposition 3.7, which are known exactly for every N, approach their limit 4κ only like $\log N/N$. The generic-plane part falls to about -10.2 and stays there for $40 \le N \le 100$, before eventually rising towards its limit (Table 5). Data with $N \le 100$ therefore cannot reveal the limit of $c_3(N)$. Section 6.4 determines it by other means.

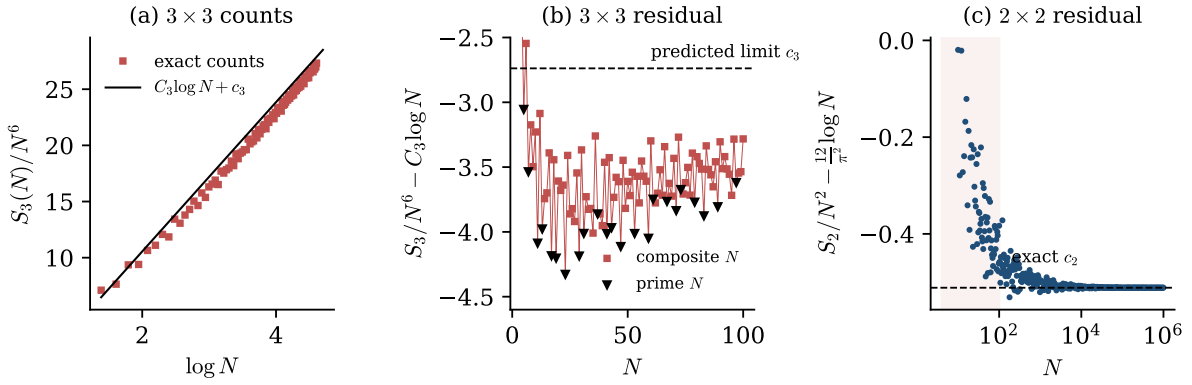


Figure 6: (a) $S_3(N)/N^6$ against $\log N$ for $N \le 100$, with the line $C_3 \log N + c_3$ using the conjectured value $c_3 = -2.738$ (Conjecture 6.1). (b) The residual $c_3(N) = S_3(N)/N^6 - C_3 \log N$. Prime values of N (triangles) sit lower than composite ones, and the dashed line is the conjectured limit. (c) The analogous 2×2 residual for $N \le 10^6$, with the exact limit c_2 of Corollary 2.4. The shaded band marks the range of panel (b).

6.3 An arithmetic effect

The residual in Fig. 6(b) depends visibly on the factorisation of N: primes sit lower than composites. The gap between the two averages shrinks steadily, from 0.57 for $20 \le N \le 29$ to 0.13 for $90 \le N \le 99$. For the explicit terms this is expected. They are sums of $\lfloor N/m \rfloor$-type expressions, which are larger when N has many divisors, but these fluctuations are of relative order $\log N/N$ by Proposition 3.7. The generic part shows a similar, decaying effect. Over $20 \le N \le 100$, about one third of the gap comes from the explicit terms and two thirds from the generic planes.

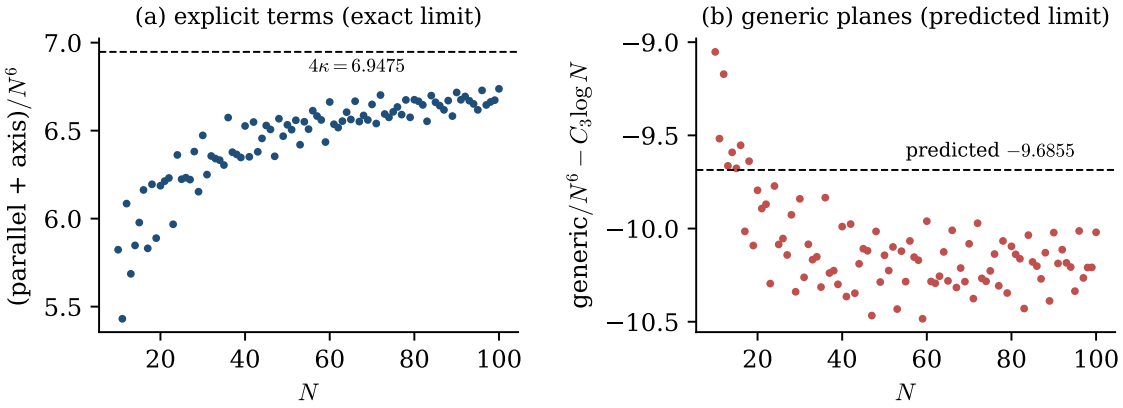


Figure 7: The two parts of $c_3(N)$ for $10 \le N \le 100$. (a) The explicit terms (parallel pairs and axis planes) divided by N^6, which converge to 4κ by Proposition 3.7. (b) The generic-plane part minus $C_3 \log N$, with the conjectured limit $c_{\text{gen}} = -9.6855$ of Conjecture 6.1.

6.4 The second-order constant

We now give a formula for $c_3 = \lim_N c_3(N)$. By Proposition 3.7, the explicit terms contribute exactly 4κ, so it suffices to find

$$c_{\text{gen}} = \lim_{N\to\infty}\Big(\frac{G(N)}{N^6} - C_3 \log N\Big), \qquad G(N) = \sum_{P \text{ generic}} L_P I_P.$$

Fix a small $\varepsilon > 0$ and split the generic planes at $|w| = \varepsilon N^2$.

Small normals, $|w| \le \varepsilon N^2$. By the estimates of Steps 2–4 in Section 5, replacing $L_P I_P$ by M_w^3 for these planes costs $O(\sqrt{\varepsilon}\, N^6) + o(N^6)$. Their contribution is therefore

$$\frac{N^6}{2}\Big(\frac{J}{\zeta(3)} \log(\varepsilon N^2) + K_0 + o(1)\Big) + O(\sqrt{\varepsilon}\, N^6),$$

where

$$K_0 = \lim_{X\to\infty}\Big(\sum_{\substack{w \text{ primitive, no zero entry} \\ |w| \le X}} \frac{\alpha(\hat{w})^3}{|w|^3} - \frac{J}{\zeta(3)} \log X\Big)$$

is the constant term in Lemma 5.4, restricted to generic normals.

Large normals, $|w| > \varepsilon N^2$. Write $\lambda = |w|/N^2$. The plane lattice Λ_w, rescaled by $N\sqrt{\lambda}$, is unimodular, and L_P counts its points in the polygon $\lambda^{-1/2} Q_{\hat{w}}$ with $Q_u = u^\perp \cap (0,1]^3$. If these rescaled lattices, together with their rotation inside the plane, become equidistributed for the Haar measure on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ as w ranges over a shell, then

$$\frac{1}{N^6} \sum_{\substack{P \text{ generic} \\ |w| > \varepsilon N^2}} L_P I_P \;\longrightarrow\; \frac{1}{2\zeta(3)} \int_\varepsilon^\infty \lambda^2\, d\lambda \int_{S^2} d\sigma(u)\, \mathbb{E}\big[F(\Lambda; \lambda^{-1/2} Q_u)\big].$$

Here Λ is a Haar-random unimodular lattice in the plane $u^\perp$, and $F = L(L^2 - \text{par})$ is computed from its points in the given polygon. Equidistribution results of this kind are known for the shapes of orthogonal lattices [1], and we test this one numerically below.

Letting $\varepsilon \to 0$, the two $\log \varepsilon$ terms cancel, because $\mathbb{E}[F] \approx \alpha(u)^3 \lambda^{-3}$ for small λ. This gives the following.

Conjecture 6.1 (Second-order constant)**.** *The limit* $c_3 = \lim_{N\to\infty}\big(S_3(N)/N^6 - C_3 \log N\big)$ *exists, and*

$$c_3 = 4\kappa + \frac{K_0}{2} + \frac{\Delta}{2\zeta(3)}, \qquad \Delta = \int_0^\infty \lambda^2\, d\lambda \int_{S^2} d\sigma(u)\Big(\mathbb{E}\big[F(\Lambda; \lambda^{-1/2} Q_u)\big] - \frac{\alpha(u)^3}{\lambda^3}\mathbf{1}_{\lambda<1}\Big).$$

Numerically, $K_0 = -7.98209$ *and* $\Delta = -13.6902 \pm 0.0015$*, so* $c_{\mathrm{gen}} = -9.6855 \pm 0.0006$ *and*

$$c_3 = -2.7381 \pm 0.0006.$$

Test in dimension two. The same derivation applies to $S_2(N)$, with lines instead of planes. There the lattice on a line has no shape, the analogue of F is $\lfloor \alpha(u)/\lambda \rfloor^2$, and the analogue of Δ can be evaluated in closed form: $\Delta_2 = 4(\gamma - \frac{1}{2} - \frac{\zeta(2)}{2}) + 2\log 2 - 4 + \pi$. Computing the lattice-sum constant numerically, $K_0^{(2)} = 0.46868230$, the formula $c_2 = K_0^{(2)}/2 + \Delta_2/(2\zeta(2))$ gives -0.5113174477. This agrees with the exact value of Corollary 2.4 to ten digits, which is the accuracy of $K_0^{(2)}$. This checks every normalisation in the derivation.

Computing K_0 and Δ. We computed the lattice sum in K_0 for $X \le 4000$ by Möbius inversion and radial binning. The error decays like $8.4/X$, and two steps of Richardson extrapolation give $K_0 = -7.98209$ to about 10^{-5}.

For Δ we used Monte Carlo, sampling the direction u uniformly in the mixed-sign orthants, the scale λ in strata, and the lattice Λ from the hyperbolic fundamental domain with a uniform rotation. The range $10^{-4} \le \lambda \le 3$ was integrated directly with about 10^9 samples, and $F = 0$ for $\lambda > 3$. For $\lambda < 10^{-4}$ we used the fitted small-λ behaviour $a + b\log\lambda$ of the integrand, which contributes -0.009.

One caution: Siegel's formula $\mathbb{E}[L] = \text{area}$ [21] suggests $3A^2(L - A)$ as a control variate, but it must not be used. For Haar-random planar lattices $\mathrm{Var}(L)$ diverges logarithmically, because of lattices with very short vectors, and the resulting estimator has unreliable error bars. The plain estimator has no such problem, since collinear points give $F = 0$.

Testing the equidistribution. Figure 8 bins the generic-plane total by λ and compares four quantities: the exact data at $N = 64$ and $N = 80$, random integer planes at $N = 10^6$, and the Haar-lattice prediction. At $N = 10^6$ the integer planes agree with the Haar prediction in every bin, within the statistical errors. The finite-N data approach the same curve from below as N grows. Replacing the Haar lattices by integer planes at $N = 10^6$ throughout $\lambda \ge 10^{-3}$ gives $c_3 = -2.7371 \pm 0.003$, consistent with the conjectured value.

Direct check at large N. Finally, we computed $c_3(N)$ itself for N up to 10^6 with a hybrid method:

- planes with $|w| \le W_0$ ($W_0 = 20$ or 30) are summed exactly, counting points with floor-function sums;
- the remaining planes are sampled by Monte Carlo over primitive normals, with the exactly known sums of M_w^3 as a control variate;

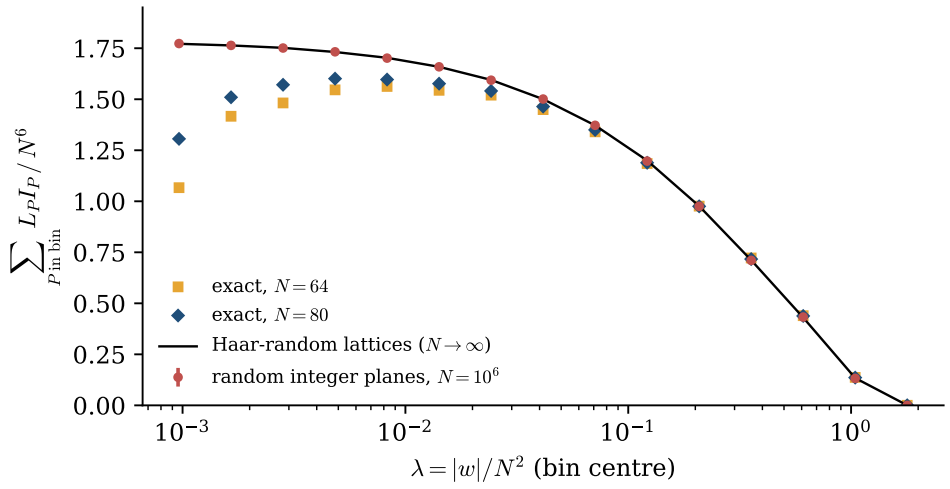


Figure 8: Contributions of generic planes with $|w|/N^2$ in logarithmic bins, from exact counts at $N = 64$ and $N = 80$, from random integer planes at $N = 10^6$ (with 2σ error bars), and from Haar-random lattices. At small λ the finite-N data fall below the curve. These bins contain only planes with small $|w|$, which are few and far from equidistributed. In the limit their contribution is captured exactly by the lattice-sum constant K_0.

- the explicit terms are computed exactly.

At $N = 100$ the method reproduces the exact count to within its error (-10.014 ± 0.009 against -10.0201 for the generic part). Table 5 shows the approach to the conjectured limit.

N	explicit terms$/N^6$	$c_{\text{gen}}(N)$	$c_3(N)$
10^2	6.73763	-10.0201 (exact)	-3.2825 (exact)
10^3	6.90773	-9.841 ± 0.008	-2.933 ± 0.008
10^4	6.94230	-9.724 ± 0.010	-2.782 ± 0.010
10^5	6.94674	-9.688 ± 0.019	-2.741 ± 0.019
10^6	6.94738	-9.645 ± 0.045	-2.698 ± 0.045
∞	$4\kappa = 6.94746$	-9.6855 ± 0.0006	-2.7381 ± 0.0006

Table 5: The second-order quantity $c_3(N) = S_3(N)/N^6 - C_3 \log N$ and its two parts. The explicit terms are exact. The generic part for $N \geq 10^3$ comes from the hybrid computation, with one-standard-deviation errors. The last row is Conjecture 6.1.

7 What remains open

Rigour for $n \geq 4$. Theorem 4.1 settles the 3×3 case, including the corner issue. For 4×4 matrices, the analogous argument would run over hyperplanes $w^{\perp} \subset \mathbb{R}^4$ and use three-dimensional sections of the cube. The lattice-point lemma then has error $O(N^2/(\mu_1\mu_2) + N/\mu_1 + 1)$, and the counting lemma bounds the number of w with small $\mu_1\mu_2$. We expect this to go through and to make Prediction 4.13 a theorem, but we have not written it out.

Second-order term. Conjecture 6.1 rests on two unproved steps. The first is the equidistribution of the rescaled plane lattices Λ_w, including their rotation, for w in a shell. The second is enough uniform integrability to exchange the limit with the sum. Equidistribution of the shapes of orthogonal lattices is known [1], and averaging over a shell of normals should make the required statement easier. A proof would establish $c_3 = -2.7381\ldots$ Closed forms for K_0 and Δ are also open. By analogy with Corollary 2.4, one might expect them to involve γ, $\zeta'(2)/\zeta(2)$, $\zeta'(3)/\zeta(3)$ and integrals of $\alpha^3 \log \alpha$.

The 4×4 constant. The closed form in Numerical identification 4.12 is not proved. Since the first two integrations are exact, a symbolic evaluation of the final one-dimensional integral, which involves logarithms and dilogarithms, would make it a theorem. It would also be interesting to know whether $J_n = 2p_n(0)$ always lies in the span of products of zeta values and $\log 2$ for $n \geq 5$, and to compute J_5 by the same method.

The sequence. The values $S_3(N)$ for $N \leq 100$ (Appendix A) appear to be absent from the OEIS and could be submitted, together with the $\{0, \ldots, N\}$ variant and cross-references to A134506 and A059306. Submission would also settle whether they are already known. The second-order constant stated in A059306 should be corrected to $3.48868\ldots$

Acknowledgments

This note grew out of an exploratory conversation with Claude (Anthropic), an AI assistant. Claude carried out the derivations, wrote and ran the counting programs, evaluated the integrals, and drafted the text under the author's direction. This includes the proof in Section 5, the determinant-density identity, the closed form of J_4, the line-by-line counting algorithm and the analysis of the second-order term. The reduction of J_4 in Section 4.6 was suggested by Gemini (Google), another AI model, which reviewed a draft. Gemini's other suggestions, an argument for the corner case and formulas for the second-order term, did not survive checking and are not used. Correcting its constants for the explicit terms led to Proposition 3.7. All formulas were checked by independent computation, as described in the text, including numerical checks of the key estimates in the proof. The proof in Section 5 was then audited a second time, line by line, by a separate instance of Claude acting as a referee, which also tested the constants in each estimate numerically. It found no gap; its corrections to constants and wording are incorporated. This is not a substitute for review by a human expert. The note has not yet been independently refereed, and any errors are the author's responsibility.

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A Exact values of $S_3(N)$

The table lists $S_3(N)$ for $N \le 100$, computed with the algorithm of Section 6. The third column is the probability $S_3(N)/N^9$ that a random matrix is singular. The last three columns give the shares of the three terms of Theorem 3.6. The values for $N \le 34$ were also computed with the independent $O(N^7)$ program, and those for $N \le 10$ by brute force.

N	$S_3(N)$	$\Pr[\det M = 0]$	parallel	axis	generic
1	1	1	1.000	0.000	0.000
2	248	4.844×10^{-1}	0.323	0.581	0.097
3	3 975	2.020×10^{-1}	0.224	0.530	0.246
4	29 104	1.110×10^{-1}	0.194	0.495	0.312
5	119 405	6.114×10^{-2}	0.164	0.445	0.390
6	436 818	4.335×10^{-2}	0.157	0.433	0.410
7	1 105 537	2.740×10^{-2}	0.142	0.400	0.458
8	2 790 818	2.079×10^{-2}	0.135	0.384	0.481
9	5 902 335	1.523×10^{-2}	0.129	0.369	0.502
10	12 074 356	1.207×10^{-2}	0.124	0.358	0.518
11	20 993 189	8.903×10^{-3}	0.117	0.341	0.542
12	40 097 796	7.771×10^{-3}	0.116	0.337	0.547
13	63 084 733	5.949×10^{-3}	0.111	0.324	0.565
14	103 877 228	5.028×10^{-3}	0.108	0.316	0.576
15	162 874 533	4.237×10^{-3}	0.106	0.312	0.582
16	252 273 262	3.671×10^{-3}	0.104	0.306	0.590
17	353 516 831	2.981×10^{-3}	0.101	0.297	0.602
18	536 248 764	2.703×10^{-3}	0.099	0.294	0.607
19	722 956 015	2.240×10^{-3}	0.097	0.286	0.617
20	1 043 346 704	2.038×10^{-3}	0.096	0.284	0.620
21	1 419 808 335	1.788×10^{-3}	0.095	0.281	0.625
22	1 916 640 106	1.588×10^{-3}	0.093	0.276	0.631
23	2 444 286 953	1.357×10^{-3}	0.091	0.270	0.639
24	3 384 786 702	1.281×10^{-3}	0.090	0.269	0.641
25	4 280 217 961	1.122×10^{-3}	0.089	0.266	0.645
26	5 508 825 848	1.015×10^{-3}	0.088	0.262	0.650
27	6 967 566 873	9.137×10^{-4}	0.087	0.259	0.654
28	8 963 655 958	8.474×10^{-4}	0.086	0.257	0.657
29	10 821 820 613	7.460×10^{-4}	0.085	0.253	0.662
30	14 023 992 516	7.125×10^{-4}	0.085	0.252	0.664
31	16 694 940 805	6.314×10^{-4}	0.083	0.249	0.668
32	20 728 595 486	5.891×10^{-4}	0.083	0.247	0.671
33	25 070 436 093	5.402×10^{-4}	0.082	0.245	0.673
34	30 304 851 862	4.991×10^{-4}	0.081	0.242	0.677
35	36 066 388 679	4.576×10^{-4}	0.081	0.241	0.679
36	44 747 190 054	4.406×10^{-4}	0.080	0.240	0.680
37	51 666 869 863	3.976×10^{-4}	0.079	0.237	0.683
38	61 166 536 772	3.702×10^{-4}	0.079	0.235	0.687
39	71 770 609 851	3.438×10^{-4}	0.078	0.233	0.689
40	86 236 338 616	3.290×10^{-4}	0.078	0.232	0.690
41	98 171 468 003	2.999×10^{-4}	0.077	0.230	0.693
42	117 541 133 292	2.890×10^{-4}	0.077	0.229	0.694
43	132 938 722 669	2.645×10^{-4}	0.076	0.227	0.697
44	155 408 100 992	2.514×10^{-4}	0.076	0.226	0.699

N	$S_3(N)$	$\Pr[\det M = 0]$	parallel	axis	generic
45	180 360 791 847	2.384×10^{-4}	0.075	0.225	0.699
46	206 853 260 248	2.243×10^{-4}	0.075	0.223	0.702
47	231 491 599 043	2.068×10^{-4}	0.074	0.222	0.704
48	272 508 172 746	2.015×10^{-4}	0.074	0.221	0.705
49	305 156 627 179	1.874×10^{-4}	0.074	0.220	0.707
50	349 834 697 264	1.791×10^{-4}	0.073	0.219	0.708
51	394 357 389 195	1.690×10^{-4}	0.073	0.218	0.710
52	449 200 441 480	1.616×10^{-4}	0.072	0.216	0.711
53	495 921 588 407	1.503×10^{-4}	0.072	0.215	0.713
54	568 800 695 124	1.457×10^{-4}	0.072	0.214	0.714
55	632 692 171 945	1.374×10^{-4}	0.071	0.213	0.715
56	718 585 632 932	1.327×10^{-4}	0.071	0.213	0.716
57	799 120 087 089	1.258×10^{-4}	0.071	0.212	0.717
58	889 933 695 820	1.198×10^{-4}	0.070	0.210	0.719
59	972 296 715 917	1.122×10^{-4}	0.070	0.209	0.721
60	1 115 724 310 608	1.107×10^{-4}	0.070	0.209	0.721
61	1 214 522 419 531	1.039×10^{-4}	0.069	0.208	0.723
62	1 343 496 572 306	9.925×10^{-5}	0.069	0.207	0.724
63	1 490 187 207 105	9.532×10^{-5}	0.069	0.206	0.725
64	1 657 497 869 398	9.201×10^{-5}	0.069	0.205	0.726
65	1 812 027 962 399	8.749×10^{-5}	0.068	0.205	0.727
66	2 025 233 224 566	8.523×10^{-5}	0.068	0.204	0.728
67	2 187 281 875 633	8.040×10^{-5}	0.068	0.203	0.729
68	2 414 113 831 388	7.766×10^{-5}	0.068	0.202	0.730
69	2 634 936 541 797	7.432×10^{-5}	0.067	0.201	0.731
70	2 918 071 417 594	7.231×10^{-5}	0.067	0.201	0.732
71	3 137 769 802 493	6.844×10^{-5}	0.067	0.200	0.733
72	3 504 316 734 108	6.739×10^{-5}	0.067	0.200	0.734
73	3 759 432 413 059	6.386×10^{-5}	0.066	0.199	0.735
74	4 088 333 532 542	6.144×10^{-5}	0.066	0.198	0.736
75	4 462 633 016 259	5.943×10^{-5}	0.066	0.198	0.737
76	4 871 472 146 044	5.759×10^{-5}	0.066	0.197	0.738
77	5 242 497 653 081	5.510×10^{-5}	0.066	0.196	0.738
78	5 756 794 871 502	5.387×10^{-5}	0.065	0.196	0.739
79	6 142 696 194 289	5.125×10^{-5}	0.065	0.195	0.740
80	6 738 040 009 304	5.020×10^{-5}	0.065	0.195	0.740
81	7 268 109 158 691	4.842×10^{-5}	0.065	0.194	0.741
82	7 834 808 187 748	4.674×10^{-5}	0.065	0.193	0.742
83	8 334 341 847 257	4.458×10^{-5}	0.064	0.193	0.743
84	9 172 932 867 060	4.405×10^{-5}	0.064	0.192	0.743
85	9 808 974 650 581	4.235×10^{-5}	0.064	0.192	0.744
86	10 536 693 809 708	4.095×10^{-5}	0.064	0.191	0.745
87	11 286 835 756 365	3.953×10^{-5}	0.064	0.191	0.746
88	12 212 864 951 764	3.859×10^{-5}	0.063	0.190	0.746
89	12 934 543 156 001	3.692×10^{-5}	0.063	0.190	0.747
90	14 137 107 892 218	3.649×10^{-5}	0.063	0.189	0.747
91	15 030 124 484 437	3.512×10^{-5}	0.063	0.189	0.748
92	16 149 128 828 126	3.420×10^{-5}	0.063	0.188	0.749
93	17 216 437 690 041	3.308×10^{-5}	0.063	0.188	0.749
94	18 378 333 143 008	3.207×10^{-5}	0.062	0.187	0.750
95	19 515 066 487 319	3.096×10^{-5}	0.062	0.187	0.751

N	$S_3(N)$	$\Pr[\det M = 0]$	parallel	axis	generic
96	21 174 354 461 568	3.058×10^{-5}	0.062	0.187	0.751
97	22 311 258 186 811	2.935×10^{-5}	0.062	0.186	0.752
98	23 852 212 046 540	2.861×10^{-5}	0.062	0.186	0.753
99	25 424 170 537 419	2.783×10^{-5}	0.062	0.185	0.753
100	27 323 920 878 598	2.732×10^{-5}	0.062	0.185	0.753

B Reproducibility

All programs and data are in the folder that accompanies this note, and each subfolder has a `README.txt` with the commands. The C programs need a compiler with OpenMP and `__int128`. The Python scripts need NumPy, SymPy, mpmath and Matplotlib.

- `count_singular_3x3.c`: the $O(N^7)$ single-N program (`gcc -O3 -fopenmp count_singular_3x3.c -o s3`, then `./s3 N`).
- `scripts/counting/`: the $O(N_{\max}^6)$ all-N program, the explicit terms, and a script that writes `singular_3x3_counts.csv`, the data of Appendix A.
- `scripts/second_order/`: the programs of Section 6.4 for K_0, Δ, random integer planes, the hybrid computation of $c_3(N)$ and the two-dimensional test. Their results are in `data/second_order_constants.txt` and `data/c3_table.csv`.
- `scripts/j4_numerics.py` and `scripts/j4_highprec/`: the quadrature for J_4 (about two minutes with 230 nodes per axis; `-check3` reproduces J_3) and the 21-digit double-double evaluation.
- `scripts/j4_semisymbolic/`: the exact inner integrations of Section 4.6 and the final quadratures. Its `identify.py` checks Numerical identification 4.12 and reruns PSLQ.
- `scripts/proof_checks/`: numerical checks of every estimate in Section 5, from the referee pass. `scripts/proof_error_term_check.py` and `scripts/explicit_terms_check.py` check Step 3 of the proof and Proposition 3.7.
- `scripts/figs_geometry.py`, `scripts/figs_numerics.py` and `scripts/rows.py`: the figures and the table.